How to get recurrence formulas?

If we can find a general expression of the solution of high-order deformation equation, in many cases, we can give recurrence formulas for the solution.

For example, let us assume that the solution of the *k*th-order deformation equation can be expressed in such a general form:

$$u_k = \sum_{n=1}^{4k+1} \alpha_{k,n} \xi^{-n} .$$

Then, from above expression, we can calculate more complicated terms, which appear in the right-hand side of the high-order deformation equation.

For example, let us consider the term

$$\xi \sum_{j=0}^{k-1} u_j \,'' \, u''_{k-1-j} .$$

Obviously, it holds

$$u_k \,'' = \sum_{n=1}^{4k+1} n(n+1) \alpha_{k,n} \xi^{-n-2} .$$

So, we have

$$\begin{aligned} & \xi \sum_{j=0}^{k-1} u_j \,'' \, u''_{k-1-j} \\ & = \xi \sum_{j=0}^{k-1} \left[\sum_{n=1}^{4j+1} n(n+1) \alpha_{j,n} \xi^{-n-2} \right] \left[\sum_{m=1}^{4k-4j-3} m(m+1) \alpha_{k-1-j,m} \xi^{-m-2} \right] \qquad (1) \\ & = \sum_{j=0}^{k-1} \sum_{n=1}^{4j+1} \sum_{m=1}^{4k-4j-3} m(m+1) n(n+1) \alpha_{j,n} \alpha_{k-1-j,m} \xi^{-(n+m+3)} . \end{aligned}$$

Define $s = m + n$. Because $1 \le n \le 4j+1,\ 1 \le m \le 4k-4j-3$, it holds

$$2 \le m + n \le 4k - 2 ,$$

i.e.

$$2 \le s \le 4k - 2 . \qquad (2)$$

Besides, $m = s - n$, thus it holds

$$1 \le s - n \le 4k - 4j - 3 .$$

From above expression, we have

$$n \le s - 1, \quad n \ge s + 3 + 4j - 4k . \qquad (3)$$

So, combining (3) with $1 \le n \le 4j+1$, we have

$$\max\{1, s+3+4j-4k\} \le n \le \min\{s-1, 4j+1\} . \qquad (4)$$

So, substituting $m = s - n$ in (1), and using (2) and (4), the expression (1) becomes

$$\xi \sum_{j=0}^{k-1} u_j\,"\,u"_{k-1-j}$$
$$= \sum_{j=0}^{k-1} \sum_{n=1}^{4j+1} \sum_{m=1}^{4k-4j-3} m(m+1)n(n+1)\alpha_{j,n}\alpha_{k-1-j,m}\xi^{-(n+m+3)}$$
$$= \sum_{j=0}^{k-1} \sum_{s=2}^{4k-2} \sum_{n=\max\{1,s+4j-4k+3\}}^{\min\{s-1,4j+1\}} n(n+1)(s-n)(s-n+1)\alpha_{j,n}\alpha_{k-1-j,s-n}\xi^{-(s+3)}$$
$$= \sum_{s=2}^{4k-2} \left[\sum_{j=0}^{k-1} \sum_{n=\max\{1,s+4j-4k+3\}}^{\min\{s-1,4j+1\}} n(n+1)(s-n)(s-n+1)\alpha_{j,n}\alpha_{k-1-j,s-n} \right] \xi^{-(s+3)}$$
$$= \sum_{s=2}^{4k-2} \beta_{k,s}\,\xi^{-(s+3)}$$

where

$$\beta_{k,s} = \sum_{j=0}^{k-1} \sum_{n=\max\{1,s+4j-4k+3\}}^{\min\{s-1,4j+1\}} n(n+1)(s-n)(s-n+1)\alpha_{j,n}\alpha_{k-1-j,s-n}\,.$$