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A new analytic algorithm of Lane–Emden type equations

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Abstract

An reliable, ease-to-use analytic algorithm is provided for Lane–Emden type equation which models many phenomena in mathematical physics and astrophysics. This algorithm logically contains the well-known Adomian decomposition method. Different from all other analytic techniques, this algorithm itself provides us with a convenient way to adjust convergence regions even without Páde technique. Some applications are given to show its validity.

Keywords: Lane–Emden equation; Theory of stellar structure; Adomian decomposition method

1. Introduction

Many problems in mathematical physics and astrophysics can be modelled by the so-called Lane–Emden type equation [1,2]

$$u''(x) + \left(\frac{2}{x}\right)u'(x) + f(u) = 0, \quad x \geqslant 0, \tag{1}$$

subject to the boundary conditions

$$u(0) = a, \qquad u'(0) = 0, \tag{2}$$

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where the prime denotes the differentiation with respect to x, a is a constant, $f(u)$ is a nonlinear function of $u(x)$. For example, it models the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [1,3,4] when $f(u) = u^m$, the gravitational potential of the degenerate white-dwarf stars [2] when $f(u) = (u^2 - C)^{3/2}$, the isothermal gas spheres [1] when $f(u) = \exp(u)$ and so on.

The difficult element in the analysis of this type of equations is the singularity behavior occurring at $x = 0$. The series solution can be found by perturbation techniques and Adomian decomposition method. However, the series solutions are often convergent in restricted regions so that some techniques such as Páde method has to be applied to enlarge the convergence regions [1,3,4].

Liao developed a kind of analytic technique for nonlinear problems, namely the homotopy analysis method [5]. Unlike perturbation techniques [6–10] and other nonperturbative methods such as the artificial small parameter method [11], the δ-expansion method [12], the decomposition method [13–31] and so on, the homotopy analysis method *itself* provides us with a convenient way to *control* the convergence of approximation series and *adjust* convergence regions when necessary. Briefly speaking, the homotopy analysis method has the following advantages:

1. it is valid even if a given nonlinear problem does *not* contain any small/large parameters *at all*;
2. it *itself* can provide us with a convenient way to control the convergence of approximation series and adjust convergence regions when necessary;
3. it can be employed to *efficiently* approximate a nonlinear problem by *choosing* different sets of base functions.

The homotopy analysis method has been successfully applied to many nonlinear problems such as viscous flows [32–35] and heat transfer [36], nonlinear oscillations [37,38], nonlinear water waves [39], Thomas–Fermi's atom model [40] and so on, and some elegant analytic results are obtained. Especially, by means of the homotopy analysis method Liao [41] gave a drag formula for a sphere in a uniform stream, which agrees well with experimental results in a considerably larger region of Reynolds number than those of *all* reported analytic drag formulas. All of these successful applications of the homotopy analysis method verify its validity for nonlinear problems in science and engineering. In this paper the homotopy analysis method is further applied to propose a reliable analytic algorithm for solving the Lane–Emden type equation and some applications are given. Our analytic approximate solutions contain Shawagfeh's [3] and Wazwaz's [4] solution given by Adomian decomposition method and besides are convergent in considerably large regions even *without* Páde technique.

2. The homotopy analysis method

2.1. Rule of solution expression

Obviously the Lane–Emden type equation can be expressed by the set of power functions

$$\mathscr{S}_1 = \{x^m | m \geqslant 0\} \tag{3}$$

such that

$$u(x) = \sum_{k=0}^{+\infty} a_k x^k, \tag{4}$$

where a_k is coefficient to be determined. This provides us with the first Rule of Solution Expression of the Lane–Emden type equation.

However, the set (3) is *not* the *unique* one to approximate the solution of the Lane–Emden type equation. Due to (1) the solution $u(x)$ decreases monotonously as x increases. So, it is possible that $u(x)$ can be approximate by the set of base functions

$$\mathscr{S}_2 = \{(1+x)^{-m} | m \geqslant 0\} \tag{5}$$

such that

$$u(x) = \sum_{k=0}^{+\infty} b_k (1+x)^{-k}, \tag{6}$$

where b_k is coefficient to be determined. This provides us with the second Rule of Solution Expression of the Lane–Emden type equation.

2.2. Choosing initial guess and auxiliary linear operator

Due to the boundary conditions (2) and the foregoing Rule of Solution Expression, it is natural to choose

$$u_0(x) = a \tag{7}$$

as the initial approximation of $u(x)$. Besides, due to (1) and the foregoing Rule of Solution Expression, it is natural to choose

$$\mathscr{L}u = u''(x) + \left(\frac{2}{x}\right) u'(x) \tag{8}$$

as the auxiliary linear operator having the property

$$\mathscr{L}\left(C_0 + \frac{C_1}{x}\right) = 0, \tag{9}$$

where C_1 and C_2 are coefficients.

2.3. Zero-order deformation equation

Let $\hbar \neq 0$ denote an auxiliary parameter, $H(x) \neq 0$ an auxiliary function, $q \in [0,1]$ an embedding parameter. Due to (1), we define the nonlinear operator

$$\mathscr{N}[\Phi(x;q)] = \frac{\partial^2 \Phi(x;q)}{\partial x^2} + \left(\frac{2}{x}\right)\frac{\partial \Phi(x;q)}{\partial x} + f[\Phi(x;q)]. \tag{10}$$

Then, we construct the zero-order deformation equation

$$(1-q)\mathscr{L}[\Phi(x;q) - u_0(x)] = q\hbar H(x)\mathscr{N}[\Phi(x;q)], \qquad q \in [0,1], \quad x \geqslant 0, \tag{11}$$

subject to the boundary conditions

$$\Phi(0;q) = a, \qquad \left.\frac{\partial \Phi(x;q)}{\partial x}\right|_{x=0} = 0. \tag{12}$$

Due to the zero-order deformation equation, it holds

$$\Phi(x;0) = u_0(x), \qquad \Phi(x;1) = u(x), \tag{13}$$

respectively. Obviously, $\Phi(x;q)$ can be expanded in the Maclaurin series of q in the form

$$\Phi(x;q) = \Phi(x;0) + \sum_{m=1}^{+\infty} u_m(x) q^m, \tag{14}$$

where

$$u_m(x) = \frac{1}{m!}\left.\frac{\partial^m \Phi(x;q)}{\partial q^m}\right|_{q=0}. \tag{15}$$

Note that the zero-order deformation equation (11) contains the auxiliary parameter $\hbar$ and the auxiliary function $H(x)$, so that $\Phi(x;q)$ is dependent upon both $\hbar$ and $H(x)$. Assuming that both $\hbar$ and $H(x)$ are so properly chosen that the series (14) is convergent when $q = 1$, one has due to (13) that

$$u(x) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x). \tag{16}$$

2.4. High-order deformation equation

Differentiating the zero-order deformation equations (11) and (12) m times with respect to q and then dividing by $m!$ and finally setting $q = 0$, we have the mth-order deformation equation

$$\mathscr{L}[u_m(x) - \chi_m u_{m-1}(x)] = \hbar H(x) R_m(x), \tag{17}$$

subject to the boundary conditions

$$u_m(0) = u'_m(0) = 0, \tag{18}$$

where

$$R_m(x) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}[\Phi(x;q)]}{\partial q^{m-1}} \right|_{q=0} \tag{19}$$

and

$$\chi_k = \begin{cases} 0, & k \leqslant 1, \\ 1, & k > 1. \end{cases} \tag{20}$$

Note that the *m*th-order deformation equations (17) and (18) are linear equations and thus can be easily solved, especially by means of symbolic software such as Mathematica, Maple, MathLab and so on.

2.5. Rule of Coefficient-Ergodicity

Due to the two different Rules of Solution Expression, the auxiliary function $H(x)$ can be either in the form

$$H(x) = x^{\alpha} \tag{21}$$

or

$$H(x) = \frac{x}{(1+x)^{\beta}}, \tag{22}$$

where α or β is coefficient to be determined by the so-called Rule of Coefficient Ergodicity, i.e. all coefficients in either (4) or (6) can be modified as the order of approximation tends to infinity. Under the Rule of Coefficient Ergodicity, our calculation indicate that, for *all* equations under consideration,

$$\alpha = 0 \tag{23}$$

for the 1st Rule of Solution Expression (4), and

$$\beta = 5 \tag{24}$$

for the 2nd Rule of Solution Expression (6).

Note that we still have the freedom to choose the value of the auxiliary parameter $\hbar$, which provides us with a convenient way to adjust the convergence region of solution series, as shown in the following section.

3. Applications

3.1. Lane–Emden equation

The thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics is modelled by the well-known Lane–Emden equation [1,3,4]

$$u''(x) + \left(\frac{2}{x}\right)u'(x) + u^m(x) = 0, \quad x \geqslant 0, \tag{25}$$

subject to the boundary conditions

$$u(0) = 1, \qquad u'(0) = 0, \tag{26}$$

where $m \geqslant 0$ is a constant.

By means of Adomian decomposition method Shawagfeh [3] and Wazwaz [4] obtained

$$u(x) = 1 + \sum_{n=1}^{+\infty} A_n x^{2n}, \tag{27}$$

where

$$A_1 = -\frac{1}{6}, \quad A_2 = \frac{m}{120}, \quad A_3 = -\frac{m(8m-5)}{3 \times 7!}, \cdots \tag{28}$$

However, for $m > 2$, (27) is not valid in the whole region with $u(x) \geqslant 0$, as shown in Figs. 1 and 2.

Under the 1st Rule of Solution Expression described by (4), we have the solution at the mth-order approximation

$$u(x) \approx 1 + \sum_{n=1}^{m} \mu_{m,n}(\hbar) A_n x^{2n}, \tag{29}$$

where the coefficients A_n are exactly the same as (28) given by Adomian decomposition method [3,4], and $\mu_{m,n}(\hbar)$ is defined by

$$\mu_{m,n}(\hbar) = (-\hbar)^n \sum_{k=0}^{m-n} \binom{n-1+k}{k} (1+\hbar)^k, \tag{30}$$

called the approaching function. Note that the convergence regions of (29) is enlarged as $\hbar$ tends to zero from below, as shown in Figs. 1 and 2. Thus, one can adjust the convergence regions of the series (29) simply by choosing a proper value of the auxiliary parameter $\hbar$.

When $\hbar = -1$ the expression (29) is the same as (27) given by Adomian decomposition method, as shown in Figs. 1 and 2. Thus, the homotopy

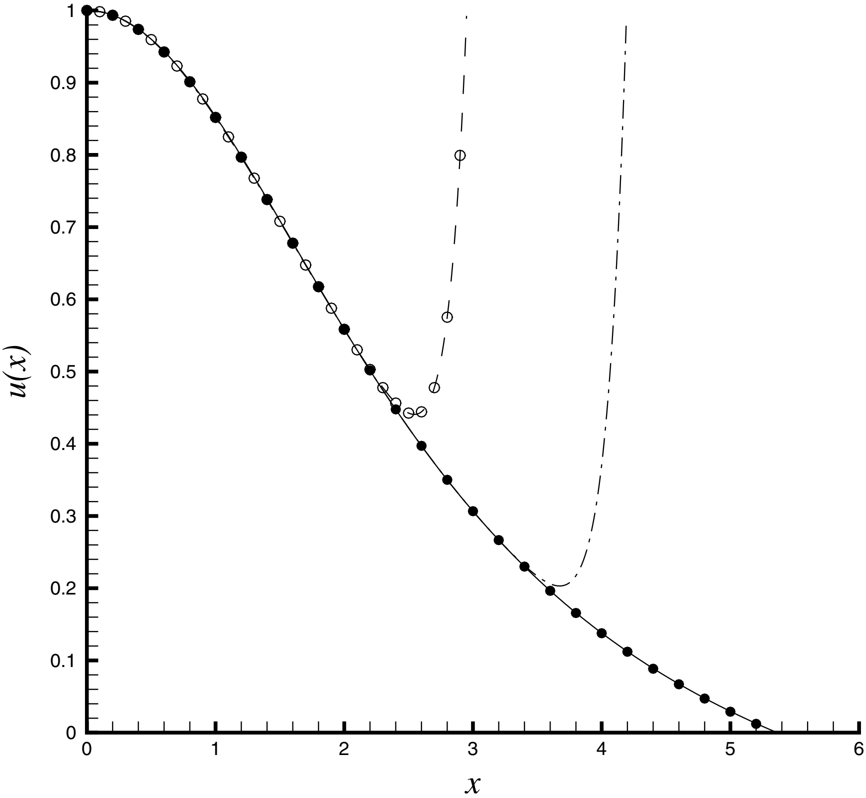

Fig. 1. Comparison of the numerical result of Lane–Emden equation when $m = 2.5$ with 10th-order analytic approximations. Filled circle: numerical result; circle: analytic result (27) given by Adomian decomposition method; dashed line: homotopy analysis approximation (29) when $\hbar = -1$; dash-dotted line: homotopy analysis approximation (29) when $\hbar = -2/3$; solid line: homotopy analysis approximation (29) when $\hbar = -1/3$.

analysis solution (29) logically contains (27) given by Adomian decomposition method [3,4].

Under the 2nd Rule of Solution Expression described by (6) we have the mth-order approximation

$$u(x) \approx \sum_{n=0}^{5m-2} \frac{\gamma_1^{m,n}}{(1+x)^n}, \tag{31}$$

where $\gamma_1^{m,n}$ is coefficient. Our calculations indicate that the above expression is convergent for $m \geqslant 0$, as shown in Fig. 3.

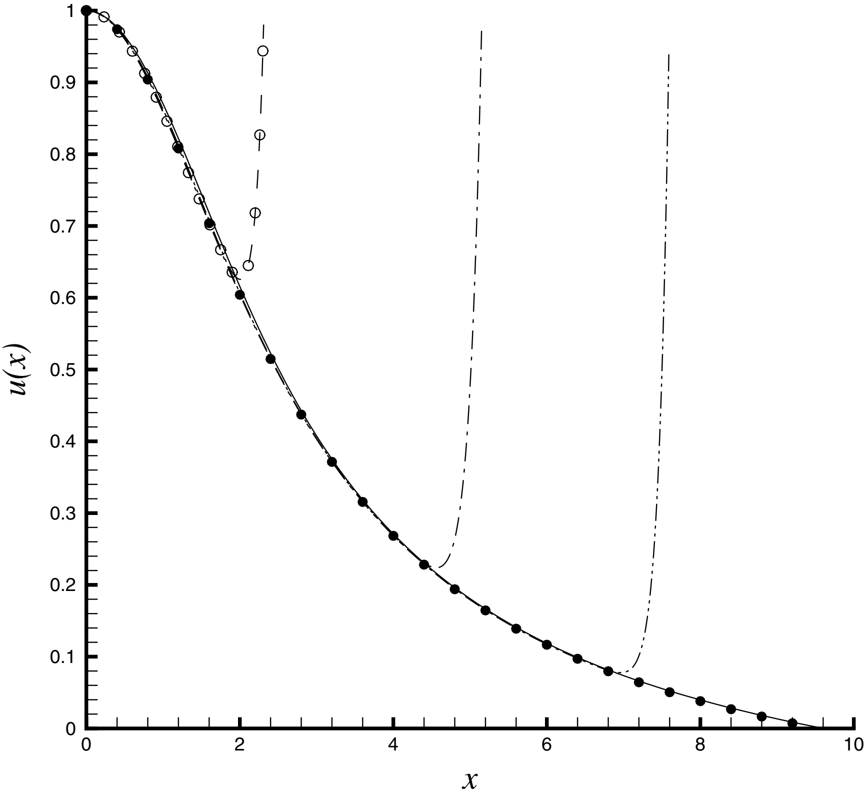

Fig. 2. Comparison of the numerical result of Lane–Emden equation when $m = 3.5$ with analytic approximations. Filled circle: numerical result; circle: analytic result (27) given by Adomian decomposition method; dashed line: 10th-order homotopy analysis approximation (29) when $\hbar = -1$; dash-dotted line: 10th-order homotopy analysis approximation (29) when $\hbar = -1/3$; dash-dot-dotted line: 16th-order homotopy analysis approximation (29) when $\hbar = -1/6$; solid line: 24th-order homotopy analysis approximation (29) when $\hbar = -1/12$.

3.2. White-dwarf equation

The gravitational potential of the degenerate white-dwarf stars can be modelled by the so-called white-dwarf equation [2]

$$u''(x) + \left(\frac{2}{x}\right)u'(x) + [u^2(x) - C]^{3/2} = 0, \quad x \geqslant 0, \tag{32}$$

subject to the boundary conditions

$$u(0) = 1, \qquad u'(0) = 0. \tag{33}$$

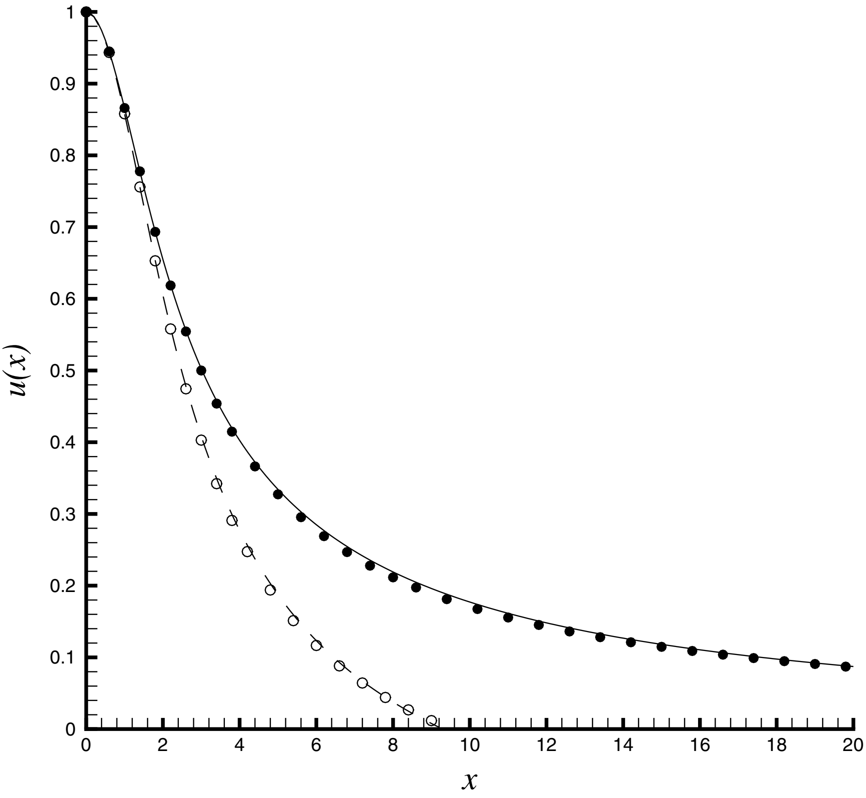

Fig. 3. Comparison of the numerical result of Lane–Emden equation with the homotopy analysis approximation (31). Circle: numerical result when $m = 3.5$; filled circle: numerical result when $m = 5$; dashed line: 20th-order homotopy analysis approximation (31) when $m = 3.5$ and $\hbar = -8$; solid line: 30th-order homotopy analysis approximation (31) when $m = 5$ and $\hbar = -6$.

By means of Adomian decomposition method Wazwaz [4] obtained

$$u(x) = 1 + \sum_{n=1}^{+\infty} B_n x^{2n}, \tag{34}$$

where

$$B_1 = -\frac{(1-C)^{3/2}}{6}, \quad B_2 = \frac{(1-C)^2}{40},$$
$$B_3 = -\frac{(1-C)^{5/2}[5(1-C)+14]}{7!}, \cdots \tag{35}$$

However, for small value of C, (34) is not valid in the whole region with $u(x) \geqslant \sqrt{C}$, as shown in Figs. 4 and 5.

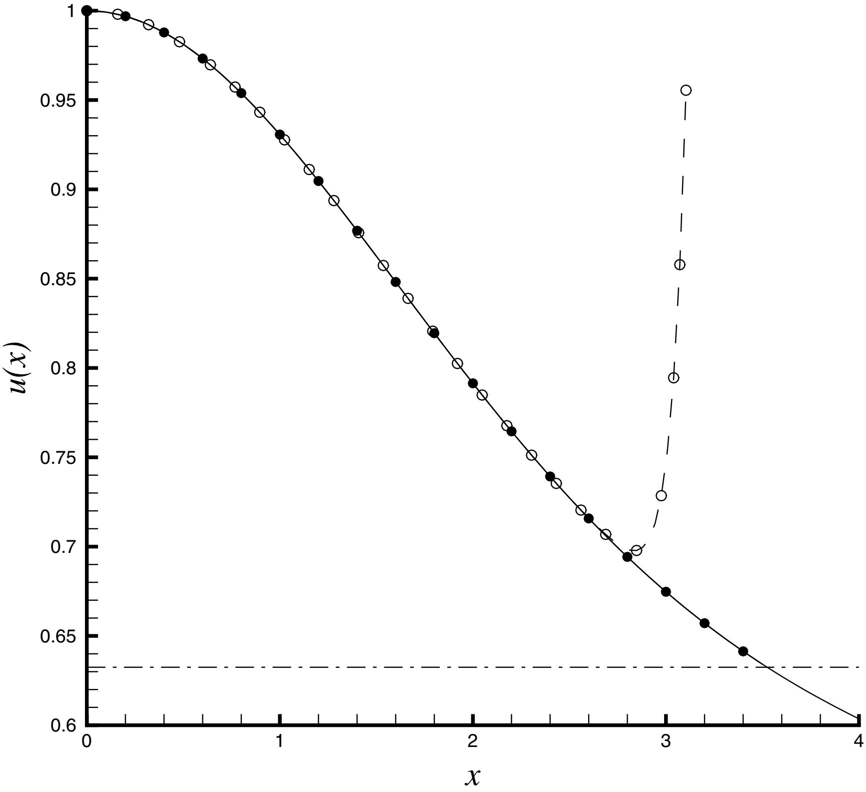

Fig. 4. Comparison of the numerical result of white-dwarf equation when $C = 2/5$ with 10th-order analytic approximations. Filled circle: numerical result; circle: analytic result (34) given by Adomian decomposition method; dashed line: homotopy analysis approximation (36) when $\hbar = -1$; solid line: homotopy analysis approximation (36) when $\hbar = -1/2$; dash-dotted line: $u(x) = \sqrt{2/5}$.

Under the 1st Rule of Solution Expression described by (4), we have the solution at the mth-order approximation

$$u(x) \approx 1 + \sum_{n=1}^{m} \mu_{m,n}(\hbar) B_n x^{2n}, \tag{36}$$

where the coefficients B_k are exactly the same as (35) given by Adomian decomposition method [4], and $\mu_{m,n}(\hbar)$ is defined by (30). Note that the convergence regions of (36) is enlarged as $\hbar$ tends to zero from below, as shown in Figs. 4 and 5. Thus, one can adjust the convergence region of the series (36) simply by choosing a proper value of the auxiliary parameter $\hbar$.

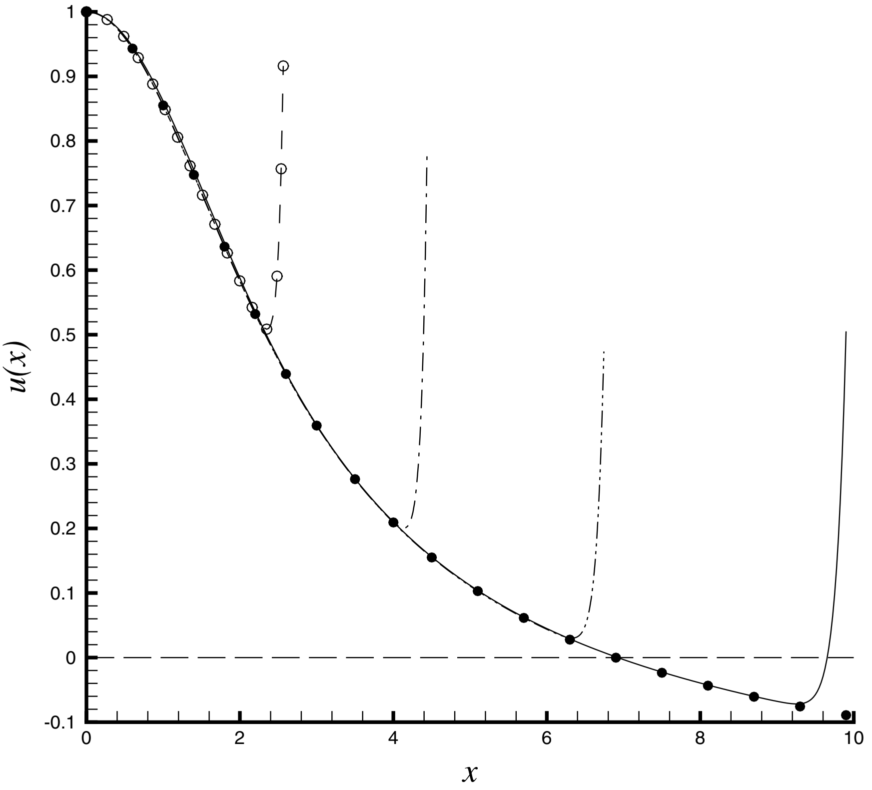

Fig. 5. Comparison of the numerical result of white-dwarf equation when $C = 0$ with analytic approximations. Filled circle: numerical result; circle: analytic result (34) given by Adomian decomposition method; dashed line: homotopy analysis approximation (36) when $\hbar = -1$; dash-dotted line: homotopy analysis approximation (36) when $\hbar = -1/2$; dash-dot-dotted line: homotopy analysis approximation (36) when $\hbar = -1/4$; solid line: homotopy analysis approximation (36) when $\hbar = -1/8$; long-dash line: $u(x) = 0$.

When $\hbar = -1$ the expression (36) is the same as (34) given by Adomian decomposition method, as shown in Figs. 4 and 5. Thus, the homotopy analysis solution (36) logically contains (34) given by Adomian decomposition method [4].

Under the 2nd Rule of Solution Expression described by (6) we have the mth-order approximation

$$u(x) \approx \sum_{n=0}^{5m-2} \frac{\gamma_2^{m,n}}{(1+x)^n}, \tag{37}$$

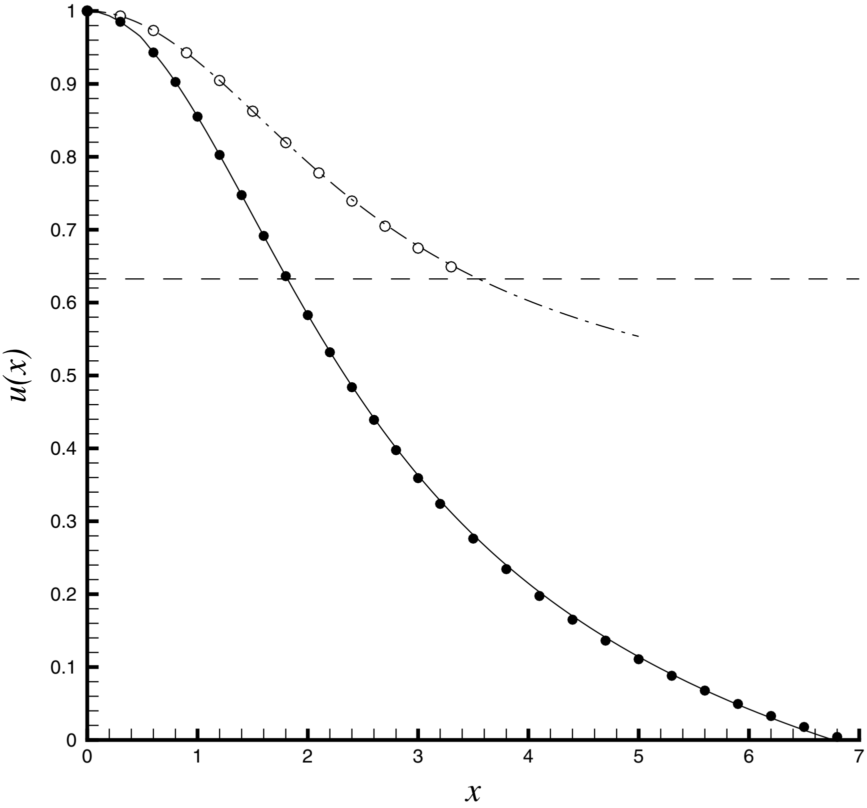

Fig. 6. Comparison of the numerical result of white-dwarf equation with the 20th-order approximation (37) when $\hbar = -10$. Circle: numerical result when $C = 2/5$; filled circle: numerical result when $C = 0$; dashed line: homotopy analysis approximation (37) when $C = 2/5$; solid line: homotopy analysis approximation (37) when $C = 0$; long-dash line: $u(x) = \sqrt{2/5}$.

where $\gamma_2^{m,n}$ is coefficient. Our calculations indicate that the above expression is convergent for $0 \leqslant C \leqslant 1$ when $-10 \leqslant \hbar < 0$, as shown in Fig. 6.

3.3. Isothermal gas spheres equation

Isothermal gas spheres [1] are modelled by

$$u''(x) + \left(\frac{2}{x}\right)u'(x) + \mathrm{e}^{u(x)} = 0, \quad x \geqslant 0, \tag{38}$$

subject to the boundary conditions

$$u(0) = 0, \qquad u'(0) = 0. \tag{39}$$

By means of Adomian decomposition method Wazwaz [4] obtained

$$u(x) = \sum_{n=1}^{+\infty} C_n x^{2n}, \tag{40}$$

where

$$C_1 = -\frac{1}{6}, \quad C_2 = \frac{1}{5 \times 4!}, \quad C_3 = -\frac{8}{21 \times 6!}, \cdots \tag{41}$$

However, (40) is valid in a rather restricted region $0 \leqslant x < 3.5$, as shown in Fig. 7.

Under the 1st Rule of Solution Expression described by (4), we have the solution at the mth-order approximation

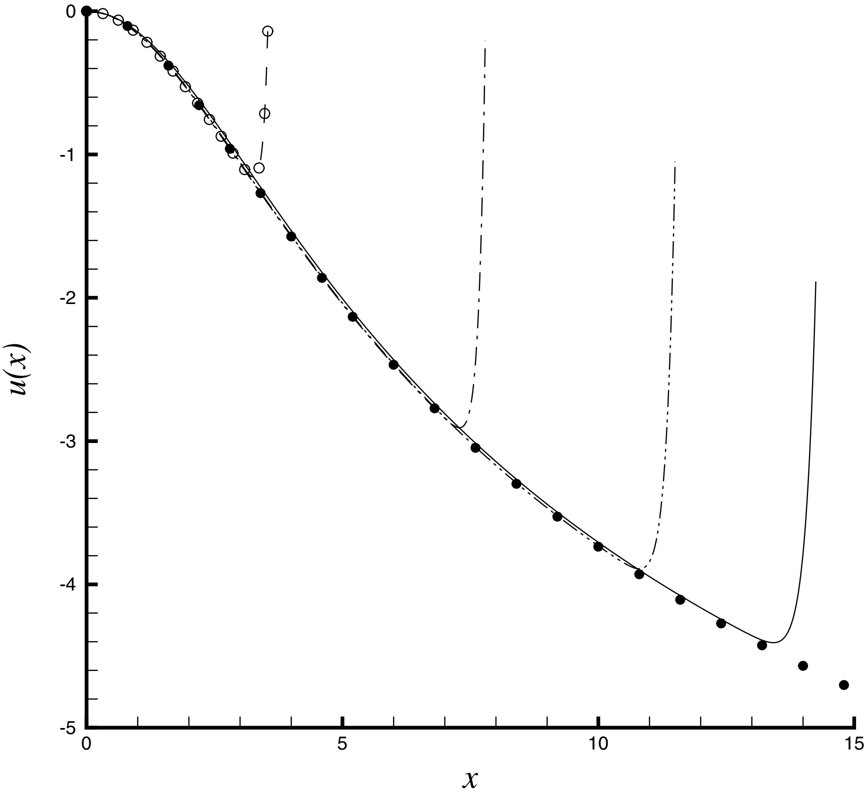

Fig. 7. Comparison of the numerical result of isothermal gas spheres equation with 20th-order analytic approximations. Filled circle: numerical result; circle: analytic result (40) given by Adomian decomposition method; dashed line: homotopy analysis approximation (42) when $\hbar = -1$; dash-dotted line: homotopy analysis approximation (42) when $\hbar = -1/3$; dash-dot-dotted line: homotopy analysis approximation (42) when $\hbar = -1/6$; solid line: homotopy analysis approximation (42) when $\hbar = -1/9$.

$$u(x) \approx 1 + \sum_{n=1}^{m} \mu_{m,n}(\hbar) C_n x^{2n}, \tag{42}$$

where the coefficients C_k are exactly the same as (41) given by Adomian decomposition method [4], and $\mu_{m,n}(\hbar)$ is defined by (30). Note that the convergence regions of (42) is enlarged as $\hbar$ tends to zero from below, as shown in Fig. 7. Thus, one can adjust the convergence regions of the series (42) simply by choosing a proper value of the auxiliary parameter $\hbar$.

When $\hbar = -1$, the expression (42) is the same as (40) given by Adomian decomposition method, as shown in Fig. 7. Thus, the homotopy analysis solution (42) logically contains (40) given by Adomian decomposition method [4].

4. Conclusions and discussions

In the frame of the homotopy analysis method an analytic algorithm is given for Lane–Emden type equation which can model many phenomena in mathematical physics and astrophysics. The analytic algorithm is reliable and ease-to-use. Its validity is verified by three examples.

First of all, our solutions (29), (36) and (42) contain the corresponding results given by Adomian decomposition method [4], thus our algorithm is more general than Adomian decomposition method. This is mainly because $\mu_{m,k}(-1) = 1$ for $0 \leqslant k \leqslant m$, as pointed out by Liao [33]. Second, different from *all* other algorithms, the convergence regions of our solutions (29), (36) and (42) can be easily adjusted by the auxiliary parameter $\hbar$, as shown in Figs. 1, 2, 4, 5 and 7. So, even *without* Páde method, our solutions (29), (36) and (42) can be valid in large enough regions. Finally, our algorithm provides two sets of different base functions (3) and (5) to approximate the solution of the Lane–Emden type equation. This provides us with the possibility to approximate solutions more efficiently, as shown in Figs. 3 and 6. All of these verify once again the validity of the homotopy analysis method and its potential in solving nonlinear problems in physics and astrophysics.

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