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On the reliability of computed chaotic solutions of non-linear differential equations

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ABSTRACT

A new concept, namely the critical predictable time T_c, is introduced to give a more precise description of computed chaotic solutions of non-linear differential equations: it is suggested that computed chaotic solutions are unreliable and doubtable when $t > T_c$. This provides us a strategy to detect reliable solution from a given computed result. In this way, the computational phenomena, such as computational chaos (CC), computational periodicity (CP) and computational prediction uncertainty, which are mainly based on long-term properties of computed time-series, can be completely avoided. Using this concept, the famous conclusion 'accurate long-term prediction of chaos is impossible' should be replaced by a more precise conclusion that 'accurate prediction of chaos beyond the critical predictable time T_c is impossible'. So, this concept also provides us a timescale to determine whether or not a particular time is long enough for a given non-linear dynamic system. Besides, the influence of data inaccuracy and various numerical schemes on the critical predictable time is investigated in details by using symbolic computation software as a tool. A reliable chaotic solution of Lorenz equation in a rather large interval $0 \le t < 1200$ non-dimensional Lorenz time units is obtained for the first time. It is found that the precision of the initial condition and the computed data at each time step, which is mathematically necessary to get such a reliable chaotic solution in such a long time, is so high that it is physically impossible due to the Heisenberg uncertainty principle in quantum physics. This, however, provides us a so-called 'precision paradox of chaos', which suggests that the prediction uncertainty of chaos is physically unavoidable, and that even the macroscopical phenomena might be essentially stochastic and thus could be described by probability more economically.

1. Introduction

One of the main goals of science is to make *reliable* predictions (Malescio, 2005). However, Lorenz (1963) found that a deterministic non-linear dynamic system might have unpredictable solutions, for example, the famous Lorenz's equations

$$\begin{cases} \dot{x}(t) = \sigma\,[y(t) - x(t)]\,, \\ \dot{y}(t) = R\,x(t) - y(t) - x(t)\,z(t), \\ \dot{z}(t) = x(t)\,y(t) + b\,z(t), \end{cases} \qquad (1)$$

where σ, R and b are physical parameters and the dot denotes the differentiation with respect to the time t, respectively, have 'non-periodic' solutions in many cases such as $\sigma = 10$, $R = 28$, $b = -8/3$, which were named 'chaos' later by Li and Yorke (1975). Chaos is a feature in all sciences (e.g. Hanski et al., 1993; Ashwin, 2003) and has the famous 'butterfly effect': solutions are exponentially sensitive to initial conditions, and thus, a tiny variation of initial conditions may bring huge difference of numerical results for a long time t.

Mostly, non-linear continuous-time dynamic systems are investigated by means of numerical integration algorithms (Parker and Chua, 1989), which model a continuous-time system by a discrete-time system. Numerical simulations are widely applied to study chaos, and such computations are often called 'numerical experiments'. Unfortunately, numerical errors are inherent in any numerical algorithms: there always exist the so-called 'numerical noise', that is, the round-off and truncation errors. For evaluating floating-point expressions, the magnitude of round-off error depends upon the hardware used. Typically, a double-precision representation uses 64 bits and is accurate to 16 decimal places. The truncation error is introduced when an infinite series is truncated to a finite number of terms. The local round-off and truncation errors propagate together in a rather complicated way, which cause the so-called global round-off error and global truncation error (Parker and Chua, 1989). So, like physical experiments, numerical experiments are also 'not' perfect.

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Exponential sensitivity to initial conditions implies that an arbitrarily small 'local' error greatly affects the macroscopic behaviour of a non-linear dynamic system with chaos, no matter whether such local error comes from the initial condition (due to the inaccuracy of measured input data) or from the 'numerical noise' mentioned above. So, not only the inaccuracy of initial conditions but also both of the round-off and truncation errors at 'each' time step eventually affect the long-term behaviour of a chaotic dynamic system. Thus, theoretically speaking, all results of chaos given by 'numerical experiments' are a kind of admixture of 'pure' solutions of non-linear dynamic systems and rather complicated propagations of the round-off error, the truncation error and the inaccuracy of initial data. Note that a lots of conclusions about chaos are based on such kind of 'inaccurate' computed results, although it has been mathematically proved that Lorenz attractor indeed exists (Stewart, 2000). Are these conclusions based on 'impure' chaotic time-series believable? Are they different from those given by the 'pure' chaotic solutions (with negligible numerical noise) if such kind of 'pure' solutions exist? Obviously, if the answers to these questions are negative, our knowledge about chaos must be changed completely.

A system of continuous-time differential equations may have various discrete-time-difference approximations with different time step τ. Each of them has different dynamic properties. It has been found (Cloutman, 1996; Cloutman, 1998) that computed results given by some discrete-time difference schemes are parasitic, which have no meanings at all. For example, when the exact time-dependent solution of a set of non-linear differential equations is known to be periodic, there is sometimes a range of the time step τ where the computed solution of the finite difference equations is chaotic (Lorenz, 1989; Cloutman, 1998). This kind of non-physical parasitic solutions is called computational chaos (CC; Lorenz, 1989). By contraries, when the exact solution is known to be chaotic, computed solutions are, however, periodic within a range of time step τ, and this numerical phenomenon is called computational periodicity (CP) (Lorenz, 2006). So, 'computed' dynamic behaviours observed for a finite time step in some non-linear discrete-time difference equations sometimes have nothing to do with the 'exact' solution of the original continuous-time differential equations at all, as pointed out by many researchers (Lorenz, 1989; Lorenz, 2006; Cloutman, 1996; Cloutman, 1998; Teixeira et al., 2007).

Lorenz (2006) investigated the influence of the time step τ on the long-term dynamic properties of a system of three non-linear differential equations:

$$\begin{cases} \dot{x}(t) = -y^2(t) - z^2(t) - Ax(t) + AF, \\ \dot{y}(t) = x(t)y(t) - Bx(t)z(t) - y(t) + G, \\ \dot{z}(t) = Bx(t)y(t) + x(t)z(t) - z(t), \end{cases} \tag{2}$$

where A, B, F and G are constant physical parameters. Using a numerical procedure based on the Mth-order truncated Taylor series in the interval $t \in [t_n, t_n + \tau]$:

$$\begin{cases} x(t) &= x(t_n) + \sum_{k=1}^{M} \alpha_k \, (t - t_n)^k, \\ y(t) &= y(t_n) + \sum_{k=1}^{M} \beta_k \, (t - t_n)^k, \\ z(t) &= z(t_n) + \sum_{k=1}^{M} \gamma_k \, (t - t_n)^k, \end{cases} \tag{3}$$

where

$$\alpha_k = \frac{1}{k!}\frac{\mathrm{d}^k x(t_n)}{\mathrm{d}t^k}, \quad \beta_k = \frac{1}{k!}\frac{\mathrm{d}^k y(t_n)}{\mathrm{d}t^k}, \quad \gamma_k = \frac{1}{k!}\frac{\mathrm{d}^k z(t_n)}{\mathrm{d}t^k}.$$

Lorenz (2006) studied the relationship between computational periodicity (CP) and the time step τ, the order M and so on. It is commonly believed that eq. (2) has chaotic solution when $A = 1/4$, $B = 4$, $F = 8$ and $G = 1$. However, when $M = 1$, the leading Lyapnuv exponent λ_1 changes sign frequently in the range $0 < \tau < \tau^*$, so that alternations between chaos ($\lambda_1 > 0$) and CP ($\lambda_1 < 0$) occur frequently. Here, τ^* is the lowest value of time step above which the computational instability (CI) occurs. As one continuously decreases the time step τ, chaos is first observed in the range $0.0402 \le \tau \le 0.0435$, then disappears, and is observed again in the range $0.0344 \le \tau \le 0.0374$, then disappears once more for the smaller τ, and is observed again when $\tau = 0.0028$, but disappears once again for the smaller τ until $\tau = 0.00039$. Rather unexpectedly, even the different chaotic solutions in case of $M = 1$ have unlike features: the intersections with plane $z = 0$ for the attractor with $\tau = 0.037$ and $\tau = 0.042$ are quite dissimilar. Similar numerical phenomena are observed for different physical parameters. Besides, when $M = 2$ or 3, the range of τ, where the CP occurs, is still much larger than the range where the true chaos is captured. Even when $M = 4$ the ranges are nearly the same. Only when $M = 6$ does the CP almost disappear. For details, please refer to Lorenz (2006).

Recently, Teixeira et al. (2007) investigated the time step sensitivity of three non-linear atmospheric models of different level of complexity, that is, Lorenz eqs (1), a quasi-geostrophic (QG) model and a global weather prediction system (NOGAPS). They illustrated that numerical convergence cannot be guaranteed forever for fully chaotic systems, because the time of decoupling of numerical chaotic solutions by different time steps follows a logarithmic rule as a function of time step for the three models. Besides, for regimes that are not fully chaotic, different time steps may lead to different model climates and even different regimes of the solution. For instance, for Lorenz eq. (1) with fully chaotic solution in case of $\sigma = 10$, $R = 28$ and $b = -8/3$, Teixeira et al. (2007) employed the same second-order numerical scheme as used by Lorenz (1963) with three different time steps: $\tau = 0.01$ (used by Lorenz, 1963), 0.001 and 0.0001 non-dimensional Lorenz time units (LTU). All solutions are quite close to each other for some initial time. However, the solution with $\tau = 0.01$ LTU decouples at about 5 LTU from the other two solutions, and the solution with $\tau = 0.001$ LTU decouples at about 10 LTU from the solution with $\tau = 0.0001$ LTU. It is interesting that all of these three solutions agree well in the interval $0 \le t \le 5$ LTU. Besides, Teixeira et al. (2007) found that the decoupling time $\hat{T}$ follows approximately $\hat{T} = \alpha - \beta \log_{10} \tau$, where $\alpha > 0$ and $\beta > 0$ are constants. Replacing τ by τ^N in this formula, where N is the order of the numerical scheme, Teixeira et al. (2007) deduced the conclusion that $\hat{T}$ should be directly proportional to N, although no direct numerical proofs were given to support it. They showed that in case of $\sigma = 10$, $b = -8/3$ and $R = 19$, the solution of $x(t)$ with $\tau = 0.01$ LTU converges to a stable positive fixed-point, while the other two solutions with $\tau = 0.001$ LTU and 0.0001 LTU converge to a stable negative fixed-point. Besides, for Lorenz equation without fully chaotic behaviour in case of $\sigma = 10$, $b = -8/3$ and $R = 21.3$, the solutions of $x(t)$ with $\tau = 0.01$ LTU and $\tau = 0.0001$ LTU converge to a stable fixed point, but the solution with $\tau = 0.001$ LTU keeps chaotic. Thus, based on these computations, they concluded that different time steps may lead to not only the uncertainty in prediction but also fundamentally different regimes of the solution. The solutions of $y(t)$ and $z(t)$ behave similarly. The same general findings mentioned above are confirmed by means of the forth-order Runge–Kutta scheme. For details, please refer to Teixeira et al. (2007).

Facing these numerical phenomena mentioned above, one might be confused: how can we ensure that a computed solution with chaotic behaviour is 'indeed' chaotic but not a so-called computational chaos (CC), and that a computed long-term solution with periodicity is 'indeed' periodic but not a computational periodicity (CP)? Unfortunately, 'exact' chaotic solutions for non-linear differential equations have never been reported. So, one even has reasons to believe that 'all chaotic responses are simply numerical noises and have nothing to do with differential equations' (Yao and Hughes, 2008a,b).

These observed numerical phenomena of the uncertainty of long-term predictions, CC and computational periodicity (CP) reveal some fundamental features of non-linear differential equations with chaos. Obviously, both CC and CP are parasitic solutions and have no physical meanings at all and, thus, should be avoided in numerical simulations. It seems that chaotic numerical results are made of reliable and unreliable parts. Also different numerical schemes might lead to completely different long-term predictions, as pointed out by Lorenz (1989, 2006) and Teixeira et al. (2007). Certainly, all conclusions based on unreliable computed results are doubtable. So, some fundamental concepts and general methods should be developed to detect the reliable solution from given computed results, which are even more important than putting forward a new numerical scheme for non-linear differential equations.

This paper is organized as follows. In Section 2, a new concept, namely the critical predictable time T_c, is introduced to detect the reliable numerical solution from calculated chaotic results. In Section 3, the influence of the round-off error, the truncation error and the inaccuracy of initial condition on the critical predictable time T_c is investigated by using Lorenz equation as an example. In Section 4, some examples are given to illustrate how computational uncertainty of prediction (CUP), CC and CP of complicated nonlinear dynamic systems can be avoided by means of the concept of the critical predictable time. In Section 5, the origin of prediction uncertainty of chaos is investigated. In Section 6, some discussions are given.

2. A strategy to detect reliable numerical results

As pointed out by Yao and Hughes (2008a), it would be an exciting contribution if 'convergent' computational chaotic solutions of non-linear differential equations could be obtained. Unfortunately, such 'convergent' solutions of chaos have never been reported. It is even unknown whether such kind of 'convergent' solutions (in traditional meaning) of chaos exist or not. Besides, it is also 'not' guaranteed whether or not a computed chaotic result obtained by the smallest time step is closest to the 'exact' chaotic solution of the continuous-time differential equations (Teixeira et al., 2007, 2008; Yao and Hughes, 2008b). How can we detect a reliable solution from different computed chaotic results? How can we avoid the so-called CC and CP?

Discovering the exponential sensitivity of chaos to initial conditions, Lorenz (1963) revealed that it is impossible to give accurate 'long-term' prediction of a non-linear dynamic system with chaotic behaviours. The current works of Lorenz (2006) and Teixeira et al. (2007) further revealed the sensitivity of computed chaotic results to various numerical schemes and different time steps. All of these current investigations confirm Lorenz's famous conclusion: accurate 'long-term' prediction of chaos is impossible (Lorenz, 1963). This conclusion is widely accepted today by scientific society. However, from mathematical points of view, this famous conclusion is not very precise, because it contains an ambiguous word 'long-term'. The concept of 'long' or 'short' is relative: 100 yr is long for everyday life but is rather short for the evolution of the universe. Is 10 non-dimensional LTU or 10^5 LTU long enough for Lorenz equation? Given a computed chaotic result, it seems that there should exist a critical time T_c^*, beyond which the computed result is unreliable or inaccurate. If the exact (or convergent) chaotic solution could be known, it would be easy to determine T_c^* simply by comparing the computed result with the exact ones. Unfortunately, no exact chaotic solutions have been reported. It is a pity that no theories about such critical time T_c^* have been proposed, so that the conclusion 'long-term prediction of chaos is impossible' is not very precise.

Lorenz (1989), Lorenz (2006) and Teixeira et al. (2007) confirmed such a numerical 'fact' that two computed chaotic results given from the same initial state by either different time steps or different numerical schemes are rather close to each other until they decouple at a critical time T_c, as illustrated in Fig. 1 for comparisons of numerical results of Lorenz's equation by means

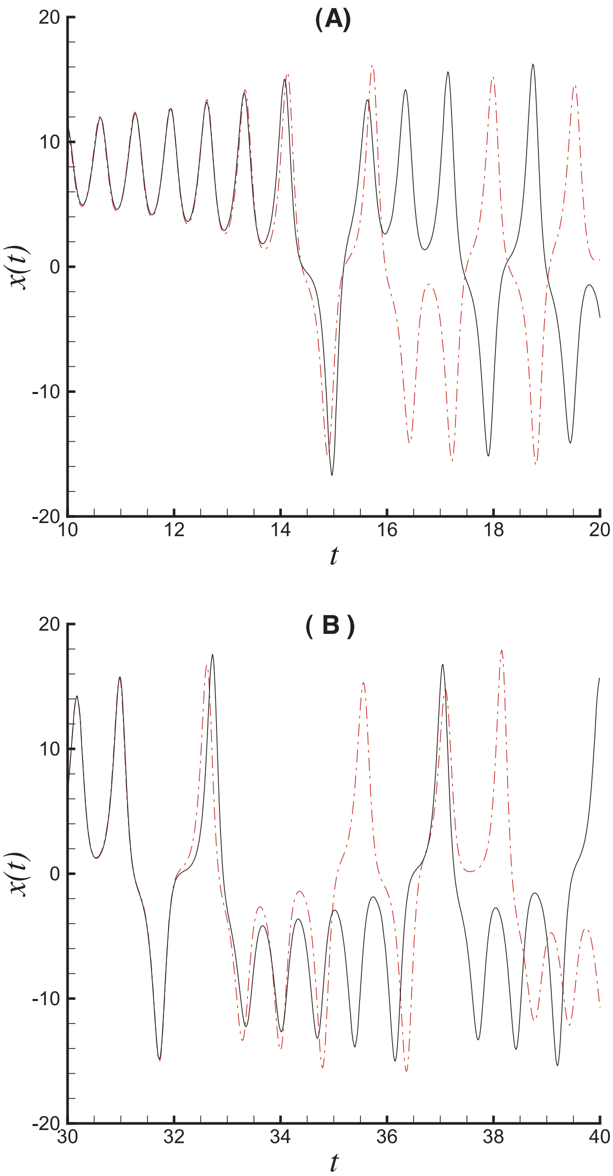

Fig. 1. Comparison of numerical results $x(t)$ of Lorenz's equation by the 4th-order Runge–Kutta's method when $\sigma = 10$, $R = 28$, $b = -8/3$ and $x(0) = -15.8$, $y(0) = -17.48$, $z(0) = 35.64$. (a) Solid line: $\tau = 10^{-5}$; dash–dotted line: $\tau = 10^{-2}$. (b) Solid line: $\tau = 10^{-5}$; dash–dotted line: $\tau = 10^{-4}$.

of the 4th-order Runge–Kutta's method with different time steps in case of $\sigma = 10$, $R = 28$, $b = -8/3$ and the initial condition $x(0) = -15.8$, $y(0) = -17.48$, $z(0) = 35.64$. Note that the numerical result given by the time step $\tau = 10^{-2}$ LTU decouples the result given by $\tau = 10^{-5}$ LTU at about 14.5 LTU, as shown in Fig. 1(a), and the result given by $\tau = 10^{-4}$ LTU decouples the result given by $\tau = 10^{-5}$ LTU at about 33.5 LTU, as shown in Fig. 1(b). Besides, Parker and Chua (1989) pointed out that a 'practical' way of judging the accuracy of numerical results of a non-linear dynamic system is to use two (or more) 'different' routines to integrate the 'same' system: the initial time interval over which the two results agree is then 'assumed' to be accurate and predictable. More precisely speaking, the computed results beyond the critical decoupling time T_c are not reliable. Here, it should be emphasized that, up to now, it is even 'not' guaranteed that the computed results in the 'whole' region $0 \leq t < T_c$ given by two different time steps or various numerical schemes are convergent or very close to the 'exact' solution, especially when the time steps are very close or the numerical schemes are rather similar. Even so, we have many reasons to 'assume' that the computed chaotic results are reliable in the region $0 \leq t < T_c$ if we properly choose two (or more[1]) different time steps and/or numerical schemes. This is mainly because the computed results in the interval $0 \leq t \leq T_c$ are 'predictable': one can get nearly the same results by different time step and/or various numerical schemes. In this way, we define a 'timescale' for the concept 'long-term' of a computed chaotic result: t is regarded to be 'long-term' if $t > T_c$. According to Lorenz (1963, 1989, 2006) and Teixeira et al. (2007), T_c is sensitive to initial condition, time step and numerical schemes used to compute the 'two' different results of the same non-linear dynamic system with chaos. For convenience, we call T_c the 'critical predictable time'. Obviously, T_c is dependent upon non-linear differential equations, time step and numerical scheme; thus, the so-called 'long-term' is also a relative concept.

The so-called 'critical predictable time' T_c can be defined in different ways. According to Teixeira et al. (2007), a numerical result given by the smallest time step is 'assumed' to be closest to the exact solution. So, Teixeira et al. (2007) defined the decoupling time by means of the state vector L2 norm error between the result obtained by the smallest time step and the result by a larger one. This kind of definition includes the error at each time step and thus is a global one for decoupling. However, the decoupling of two curves is essentially a local occurrence. Thus, we give here a local definition of 'critical predictable time' T_c, which is based on geometrical characteristic of decoupling of two curves and thus is straightforward. Mathematically, let $u_1(t)$ and $u_2(t)$ denote two time-series given by different numerical routines for a given dynamic system. The so-called 'critical predictable time' T_c for $u_1(t)$ and $u_2(t)$ is determined by the criteria

$$\dot{u}_1\, \dot{u}_2 < -\epsilon, \quad \left|1 - \frac{u_1}{u_2}\right| > \delta, \quad \text{at } t = T_c, \tag{4}$$

where $\epsilon > 0$ and $\delta > 0$ are two small constants (we use $\epsilon = 1$ and $\delta = 5\%$ in this paper). Mathematically, the critical predictable time T_c can be interpreted as follows: the influence of truncation error, round-off error and inaccuracy of initial condition on numerical solutions is negligible in the interval $0 \leq t \leq T_c$, so that the computed result is predictable and, thus, can be regarded as a reliable solution in this interval. Using the concept

[1] Obviously, it is better to compare computed results given by disparate numerical schemes with different time steps: the more, the better.

of the critical predictable time T_c, the famous statement that 'accurate long-term prediction of chaos is impossible' can be more precisely expressed as that 'accurate prediction of chaos beyond the critical predictable time T_c is impossible'. Here, T_c is regarded as a critical point: computed results beyond the critical predictable time T_c are doubtable and unreliable. Thus, the critical predictable time T_c provides us a strategy to detect the reliable solution from a given numerical result.

As pointed out by Lorenz (1989, 2006), CC and CP are mainly based on the evaluation of Lyapnunov exponent, which is a long-term property. As mentioned above, any computed results for $t > T_c$ are doubtable and unreliable and, thus, have no meanings. Unfortunately, most of computed 'long-term' solutions are often far beyond the critical predictable time T_c, and thus, all related conclusions or calculations based on these doubtable 'long-term' numerical results, such as CC, CP, Lyapunov exponent and attractors, are unreliable, too. Note that, using the concept of the critical predictable time T_c, the third figure given by Teixeira et al. (2007) should be interpreted in such a new way: the critical predictable time T_c for three computed results given, respectively, by $\tau = 0.01$, 0.001 and 0.0001 LTU is less than 15 LTU; so, all computed results beyond $t > 15$ LTU have no meanings, and thus, one 'cannot' make such a conclusion that 'different time step may not only lead to uncertainty in the predictions after some time, but also lead to fundamentally different regimes of the solution' (Teixeira et al., 2007). In fact, by means of the concept of the critical predictable time, the CUP, CC and CP can be avoided completely, as shown in Section 4 for details.

As suggested by Parker and Chua (1989), all numerical results should be interpreted properly. The critical predictable time T_c can be understood as follows: the influence of truncation error, round-off error and inaccuracy of initial condition on computed chaotic solutions is almost negligible in the time interval $0 \leq t \leq T_c$. Thus, the so-called critical predictable time T_c provides us a scale to investigate chaos in a more precise way. This new concept may greatly deepen and enrich our understanding about chaos, not only mathematically but also physically, as shown later.

3. Influence of numerical scheme and data inaccuracy on T_c

Since computed chaotic results beyond the critical predictable time T_c are unreliable, a numerical solution with small T_c is almost useless. Thus, it is necessary to obtain reliable chaotic solutions with large enough T_c.

Without loss of generality, let us consider Lorenz's eq. (1) in case of $\sigma = 10, R = 28, b = -8/3$, with the exact initial condition $x(0) = -15.8, y(0) = -17.48, z(0) = 35.64$. Using the 4th-order Runge–Kutta's method with different time increment $\tau = 10^{-2}$, 10^{-3}, 10^{-4} and 10^{-5} LTU, the corresponding critical predictable times of computed chaotic results are about 13.7, 24.5 and 32.6 LTU, respectively, as shown in Fig. 1. So, by means of traditional numerical methods, the critical predictable time T_c of computed chaotic results is often not long enough. Teixeira et al. (2007) found that for given numerical scheme, the decoupling time of numerical chaotic results with different time steps follows a logarithmic rule as a function of time step τ. Thus, the time step should be exponentially small for a given critical predictable time T_c. Lorenz (2006) reported, qualitatively, some influences of numerical schemes (based on the truncated Taylor series at a few different orders M) on the computed chaotic results but did not give a quantitative relationship between the critical predictable time T_c and the approximation order M. Besides, it is a pity that the influence of the round-off error on the decoupling time of computed chaotic results given by various numerical schemes is neglected, mainly because traditional floating-point computations use data in either single or double precision only.

Currently, some advanced symbolic computation software, such as MATHEMATICA and MAPLE, are widely used. In this paper, the symbolic computation software MATHEMATICA (Wolfram Research, Champaign, IL) is employed as a computational tool to investigate the influence of various numerical schemes based on the truncated Taylor series (3), the round-off error and the inaccuracy of initial condition on the critical predictable time T_c. From the view-point of round-off error, symbolic computation is completely different from evaluating floating-point expressions: the round-off error can be almost neglected or even 'avoided' by means of symbolic computation. For example, by means of symbolic computation, we can have the 'exact' result $1/2 + 1/3 = 5/6$. Note that, using numerical computation with double precision representations, one has only the 'approximate' result $1/2 + 1/3 \approx 0.8333333333333333$, whose round-off error is about 10^{-16}. Besides, using the command 'N[Pi, 800]' of MATHEMATICA, we can get the approximation $\pi \approx 3.14159265358979323846264338 32 \cdots$, which is accurate even to 800 decimal places! By means of so precise data representation provided by symbolic computation software, the round-off error can be almost neglected. Let K denote the number of decimal places of all data used by the symbolic software in this paper. Then, by means of different values of K, it is easy to investigate the influence of the round-off error on T_c, as shown later. Furthermore, by means of the truncated Taylor series scheme (3), the system of Lorenz equations (1) is approximated by a time-continuous system in each interval $t \in [t_n, t_n + \tau]$ as the truncated Mth-order Taylor's expansion. Obviously, the truncation error of this scheme is determined by M. Therefore, using symbolic computation and the analytic approach described above, it is convenient to control the magnitude of the truncation and round-off errors by means of M and K, respectively. Clearly, the larger the values of M and K, the smaller the truncation error and the round-off error, respectively. Thus, the symbolic computation software provides us a useful tool to investigate the influence of truncation-error, round-off error and inaccuracy of initial conditions on the critical predictable time T_c.

Without loss of generality, we consider here Lorenz eq. (1) in case of $\sigma = 10$, $R = 28$, $b = -8/3$ with the initial condition $x(0) = -15.8$, $y(0) = -17.48$, $z(0) = 35.64$ and the time step $\tau = 10^{-2}$, if not mentioned particularly. Note that the initial condition is assumed to be exact. To investigate the influence of the truncation error on computed chaotic results alone, we set a large enough number of decimal places, that is, $K = \max\{200, 2M\}$, where M is the order of truncated Taylor series (3) of Lorenz's eq. (1). In this way, the round-off error is much smaller than the truncation error and, thus, is negligible. Since the initial condition is assumed to be exact, there exists truncation error alone, whose magnitude is determined by M, the order of truncated Taylor series (3) of Lorenz eq. (1). Using different values of M from $M = 4$ to 110, we get different computed results with different truncation errors. Using (4) as the decoupling criteria of two computed trajectories, it is easy to find the corresponding critical predictable time T_c of the numerical result given by the smaller M. It is found that the critical predictable time T_c is directly proportional to M, that is,

$$T_c \approx 3M, \tag{5}$$

as shown in Fig. 2.

It is a little more difficult to investigate the influence of the round-off error on chaotic results alone, mainly because the round-off error might greatly increase for given K, when the order M is too large compared with K. Note that the previous formula $T_c \approx 3M$ (with $K = \max\{200, 2M\}$) gives a time interval $0 \le t \le T_c$ in which the influence of both truncation error and round-off error is negligible, as interpreted before. For example, when $M = 32$ and $K = 200$, the influence of the truncation error and the round-off error is negligible for $t \le 96$. Thus, without loss of generality, let us consider the case of $M = 32$, with different values of K ($K < 100$). Comparing the results given by different values of K ($K < 100$) with the result obtained by $K = \max\{200, 2M\}$, we get the corresponding critical predictable times T_c. It is found that, when $K > 40$, T_c tends to the same value close to 96, respectively. This is mainly because, when K is large enough, the round-off error is much smaller than the truncation error. So, the results for $K > 40$ is useless in investigating the influence of the round-off error on T_c. It is also found that, when $K \le 16$, the precision of computation is too low relative to the M, the order of approximation, so that the round-off error increases greatly. Thus, the results with too small K is also useless. So, only results given by proper values of K are useful. It is found that, for $18 \le K \le 40$, the computed critical predictable times agree well with the formula

$$T_c \approx 2.51K - 4.26, \tag{6}$$

as shown in Fig. 3. Furthermore, it is found that, in general, the critical predictable time T_c indeed increases 'linearly' with respect K, the number of accurate decimal places of results.

Note that the initial conditions are assumed to be exact in above computations. According to our above investigations, in case of $K = 200$ and $M = 100$, both of the round-off error and the truncation error are negligible in the interval $0 \le t \le 300$. This provides us a convenient way to investigate the influence of the inaccuracy of initial conditions on T_c alone. To do so, we add a tiny difference Δx_0 into the initial

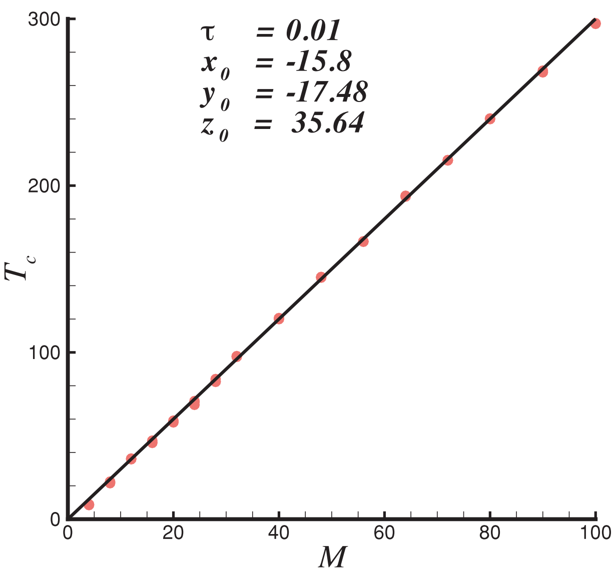

Fig. 2. The critical predictable time T_c versus the order M of truncated Taylor series (3) in case of $\sigma = 10$, $R = 28$, $b = -8/3$ and $x(0) = -15.8$, $y(0) = -17.48$, $z(0) = 35.64$ with $K = \max\{200, 2M\}$. Symbols: computed results; Solid line: $T_c = 3M$.

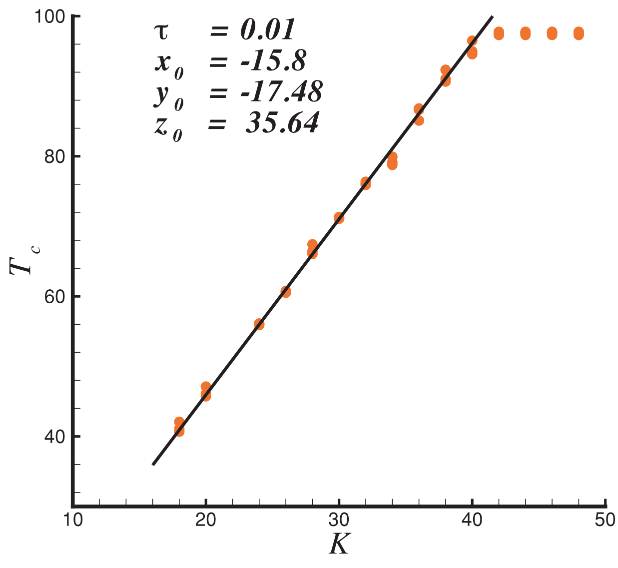

Fig. 3. The critical predictable time T_c versus K (the number of accurate decimal places) in case of $\sigma = 10$, $R = 28$, $b = -8/3$ and $x(0) = -15.8$, $y(0) = -17.48$, $z(0) = 35.64$ with $M = 32$. Symbols: computed results; Solid line: $T_c = 2.51\,K - 4.26$.

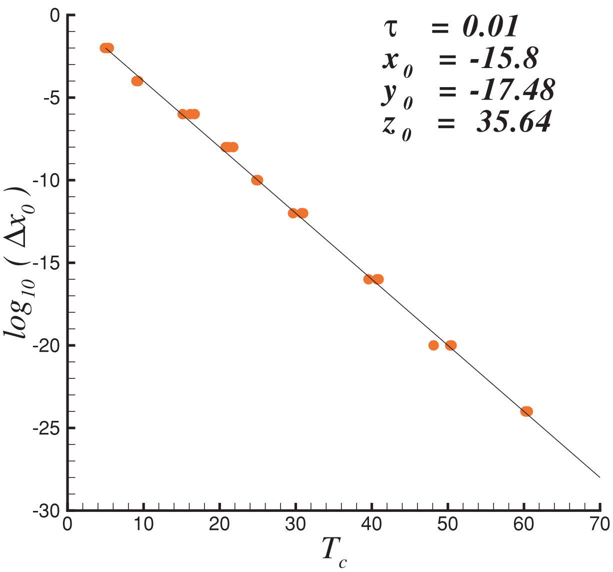

Fig. 4. The critical predictable time T_c versus the inaccuracy Δx_0 of initial conditions in case of $\sigma = 10$, $R = 28$, $b = -8/3$ and $x(0) = -15.8 + \Delta x_0$, $y(0) = -17.48$, $z(0) = 35.64$ with $M = 100$ and $K = 200$. Symbols: computed results; Solid line: $\log_{10}(\Delta x_0) = -0.4\, T_c$.

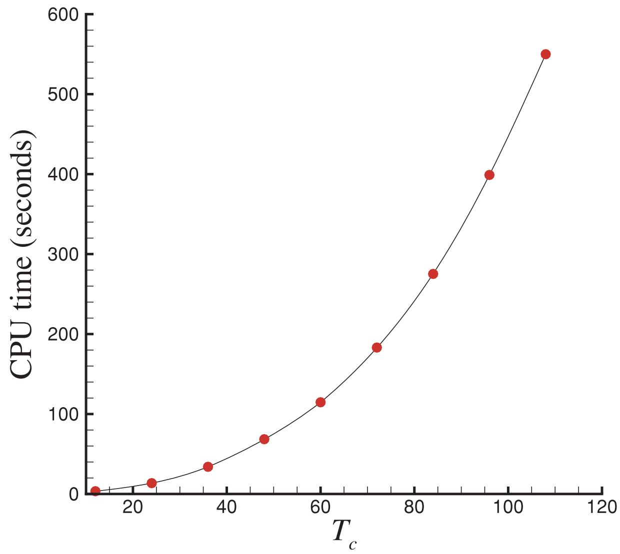

Fig. 5. The used CPU time versus T_c in case of $\sigma = 10$, $R = 28$, $b = -8/3$ and $x(0) = -15.8$, $y(0) = -17.48$, $z(0) = 35.64$ with $M = T_c/3$ and $K = 200$.

condition $x(0) = -15.8$, $y(0) = -17.48$, $z(0) = 35.64$ in such a way that

$$x(0) = -15.8 + \Delta x_0,$$

but with the same values of $y(0)$ and $z(0)$. Comparing the results given by different values of $\Delta x_0 > 0$, $M = 100$ and $K = 200$ with the result given by $\Delta x_0 = 0$, $M = 100$ and $K = 200$, we obtain the corresponding T_c by means of the decoupling criteria (4). It is found that for given different values of Δx_0, the corresponding results of T_c agree well with the formula

$$T_c \approx -2.5 \log_{10}(\Delta x_0),$$

as shown in Fig. 4. This formula can be rewritten as

$$\Delta x_0 \approx 10^{-0.4\, T_c}, \qquad (7)$$

which means that the precision of the initial condition must increase 'exponentially' with respect to a given critical predictable time T_c. For example, to get a reliable chaotic solution with $T_c = 200$ LTU, the initial condition must be with the precision $\Delta x_0 < 10^{-80}$. Therefore, we need a rather precise initial condition to get a reliable chaotic solution with $T_c > 200$ LTU. Unfortunately, such precise initial conditions are impossible in practice, as discussed in Section 5. That is exactly the reason why the 'butterfly effect' exists, as pointed out by Lorenz (1963). However, the formula (7) might inform us much more than the so-called 'butterfly effect', as discussed in details in Section 5.

Can we get reliable chaotic solutions with large enough critical predictable time T_c? Assuming that the initial condition is exact, it is found that $T_c \approx 3\, M$ generally holds in case of $K = \max\{200, 2M\}$. Therefore, M (the order of truncated Taylor series) and K (the number of accurate decimal places) should be increased linearly with respect to the critical predictable time T_c. So, theoretically speaking, for a given T_c, one can always find the corresponding value of M and K to get a 'reliable' chaotic solution in $0 \le t \le T_c$. However, the CPU time increases with respect to T_c in a power-law, as shown in Fig. 5. Suppose that we would like to get a reliable chaotic solution with $T_c = 1200$ LTU. According to (5), we should choose $M = 400$ so as to get such a reliable chaotic result. In fact, by means of $M = 400$ and $K = 800$, we indeed obtain such a reliable chaotic solution in the interval $0 \le t < 1200$ LTU, as shown in Figs 6 and 7. The corresponding result is rather precise: the maximum residual error is only 1.1×10^{-481}. However, more than 461 h 16 min CPU time (more than 19 d) is used by a cluster Intel Clovertown Xeon E5310 with 8GB RAM. To the best of our knowledge, such kind of reliable chaotic solution of Lorenz equation in such a long time interval has never been reported. Based on this time-consuming computation, we are quite sure that the solution of Lorenz eq. (1) in case of $\sigma = 10$, $R = 28$ and $b = -8/3$ is indeed chaotic 'within' the interval $0 \le t < 1200$ LTU, as shown in Table 1.[2] However, strictly speaking, it is unknown whether or not the chaotic behaviour disappears when $t > 1200$ LTU. This is because, based on our current computations, chaotic numerical results beyond T_c is unreliable. To answer this question, one has to spend more CPU time to get a reliable chaotic solution

[2] The full data may be downloaded in the online version of this paper (Supporting Information) or on the author's website: http://numericaltank.sjtu.edu.cn/.

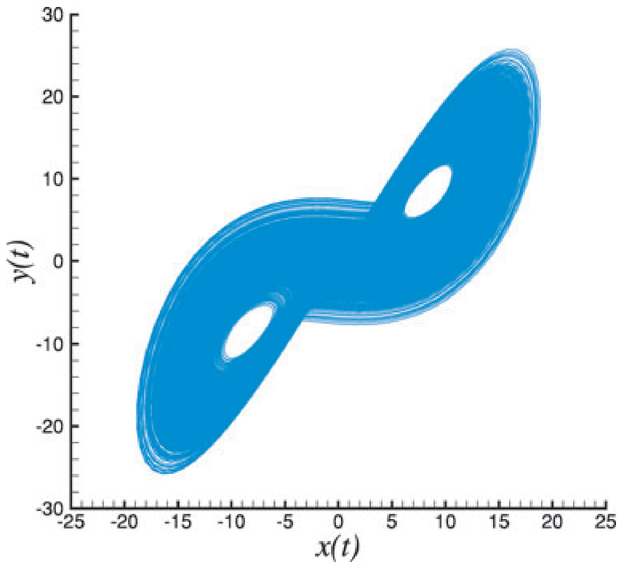

Fig. 6. The curve $x(t)$–$y(t)$ given by the reliable chaotic result with $T_c = 1200$ LTU by $\tau = 0.01$, $M = 400$, $K = 800$ in case of $\sigma = 10$, $R = 28$, $b = -8/3$ and $x(0) = -15.8$, $y(0) = -17.48$, $z(0) = 35.64$.

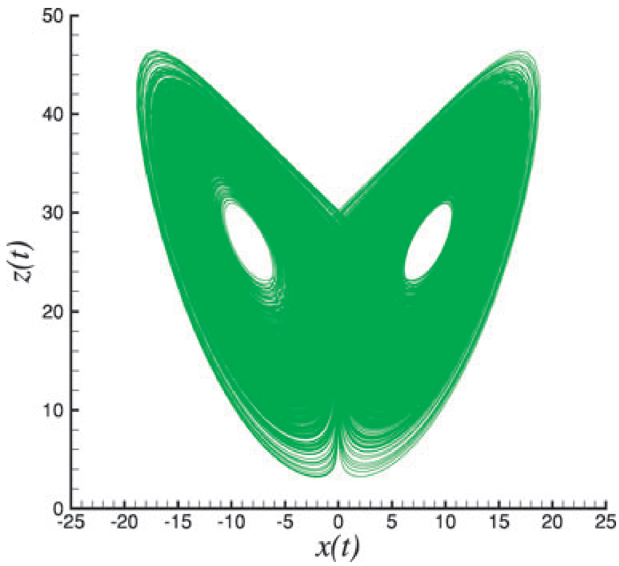

Fig. 7. The curve $x(t) - z(t)$ given by the reliable chaotic result with $T_c = 1200$ LTU by $\tau = 0.01$, $M = 400$, $K = 800$ in case of $\sigma = 10$, $R = 28$, $b = -8/3$ and $x(0) = -15.8$, $y(0) = -17.48$, $z(0) = 35.64$.

with even larger T_c. Unfortunately, T_c is always a finite value, no matter how large it is! Also the non-linearly increased CPU time indicates the impossibility to get a reliable chaotic solution in an infinite interval $0 \le t < +\infty$. This is revealed from the view-point of CPU time that chaos is mathematically unpredictable in essence.

4. Avoidance of computational chaos, computational periodicity and computational uncertainty of prediction

The uncertainty of prediction of chaos have two reasons. One is computational (or more precisely, mathematical), which is due to non-linearity of models and the imperfection of numerical schemes and data precision mentioned in Section 3. The other is physical, which is based on the fundamental physical principles of nature.

In this section, we investigate the computational uncertainty (CU) of chaos. In essence, the CU of long-term prediction comes from the unpredictability of trajectories, that is, the decoupling of different trajectories for a long time. By means of the concept of the critical predictable time T_c and regarding chaotic results unreliable when $t > T_c$, the numerical phenomena such as CC, CP and CU, can be avoided completely, as illustrated below.

It is well known that solution of Lorenz's equation (1) becomes unstable if $R > R_c = \sigma(\sigma - b + 3)/(\sigma + b - 1)$. In case of $\sigma = 10$ and $b = -8/3$, we have the critical value $R_c = 24.7368$. Thus, in case of $\sigma = 10$, $b = -8/3$ and $R = 19 < R_c$ with the initial state $x = y = z = 5$, the exact solution should tend to a fixed point. However, it is unknown which fixed point the solution tends to. It is found that the computed result $x(t)$ given by the Mth-order scheme (3) with $\tau = 0.01$ LTU tends to a 'negative' fixed-point for $M = 2$ but goes to a 'positive' fixed point for $M = 3$, as shown in Fig. 8. Thus, at least one of these two different predictions must be wrong. However, based on these two computational results, it is hard to detect which prediction is correct. This kind of CUP is similar to those mentioned by Teixeira et al. (2007). Note that the critical predictable time T_c of these two computed results are only about $T_c \approx 9$ LTU, as shown in Fig. 8, so that they are reliable only in the interval $0 \le t < 9$ LTU. In other words, the two computed results in the interval $t > 9$ are 'unreliable' and, thus, has no meanings. Therefore, based on these two computational results, one cannot give any reliable conclusions about the fixed point. To get a reliable prediction about the fixed-point, we had to give a 'reliable' solution with large enough T_c. To do so, we use the truncated Taylor series scheme (3) with much higher order M. As shown in Fig. 9, the two computed results given by $M = 30$ and 40 with $\tau = 0.01$ LTU agree well in the interval $0 \le t \le 100$ LUT, and both of them give the same' numerical fixed-point:

$$x(100) = -6.928204, \quad y(100) = -6.928204,$$

$$z(100) = 18.000000.$$

Based on these two reliable results, we are quite sure that the exact solution $x(t)$ of Lorenz equation (1) tends to a negative fixed point in case of $\sigma = 10$, $b = -8/3$ and $R = 19 < R_c$ with the initial state $x = y = z = 5$. In this way, the CUP mentioned by Teixeira et al. (2007) and Lorenz (2006) can be avoided.

Similarly, the so-called CC and CP mentioned by Lorenz (1989, 2006) can be turned away, too. For example, when $\sigma = 10$,

Table 1. Some reliable numerical results with $T_c = 1200$ LTU in case of $\sigma = 10$, $R = 28$, $b = -8/3$ by means of $M = 400$, $K = 800$ and $\tau = 0.01$

0	−15.8	−17.48	35.64
50	12.779038299490452	8.825054357006032	36.40092236534542
100	−10.510118721506247	−12.17254281368225	27.476265630374762
150	−1.9674157212680177	−2.5140404626072206	17.233128197642884
200	−6.697233173381982	−11.911020483539128	13.036826414358321
250	3.480010996527037	5.743865139093177	22.424028925951887
300	10.197534991661733	3.906517722362926	35.33742709240441
350	0.009240166388150502	−1.1520585946848019	20.259118270313508
400	−1.8892476498049868	−3.5657880408974663	20.299639635504597
450	2.3442055460290803	2.473910407011588	19.324756580383077
500	−5.30509963157152	−9.425991029211517	12.302184230689779
550	−9.710817000847529	−6.878169205988265	31.67393963382737
600	−0.8635053825976141	0.499057856286716	21.581438144249077
650	−6.249196468824656	−1.3133350412836564	30.3936296733578
700	10.884963668216704	16.32989379246704	22.247458859587212
750	−1.5200586402319973	−0.4164281272461717	21.530357757012936
800	1.3963347154139534	2.40877126758134	14.590441270059282
850	1.580132807298193	2.6273272210193146	12.83939375621528
900	−6.449367823985297	−10.984642417532422	14.647974468278282
950	10.098469202323805	0.4959032511015884	37.72812801085503
1000	13.881997000862393	19.918303160406396	26.901943308376104
1025	−2.831908677750036	−5.127291386139972	10.787422525560384
1050	−6.0495817084397405	−0.5249599507390699	30.805747242725836
1075	−8.445628564097573	−16.91583633884055	8.185099340204886
1100	2.2974592711836634	2.299710874996516	19.617779431769037
1125	−2.0420317363264457	−0.3357510158682992	23.174657463445286
1150	−14.378782424952437	−11.819346602645444	37.319351169225996
1175	−11.794511899005188	−13.181679857519981	29.65720151904728
1200	2.4537546196402595	4.124943247158509	19.349201739150004

Note: The full data may be downloaded in the online version of this article (supporting information) or on the author's website: http://numericaltank.sjtu.edu.cn/

$b = -8/3$ and $R = 21.5 < R_c$ with the initial state $x = y = z = 5$, it is found that the computed result $x(t)$ given by the Mth-order truncated Taylor series scheme (3) with $\tau = 0.01$ LTU keeps chaotic when $M = 2$ but tends to a positive fixed-point when $M = 3$, as shown in Fig. 10. Since $R = 21.5 < R_c$, the exact solution of $x(t)$ must tend to a fixed point in a large enough time, therefore the chaotic solution given by $M = 2$ is obviously wrong, although it is unknown whether the exact solution of $x(t)$ indeed tends to the 'positive' fixed point or not. To get a correct prediction, reliable results with large enough critical predictable time T_c are needed. It is found that the computed result given by $M = 30$ agrees well with that given by $M = 40$ in the interval $0 \le t \le 100$, as shown in Fig. 11. These two results, which are reliable in $0 \le t \le 100$, clearly indicate that the solution in case of $\sigma = 10, b = -8/3$ and $R = 21.5 < R_c$ with the initial state $x = y = z = 5$ is 'not' chaotic, and besides, $x(t)$ tends to a 'negative' fixed-point. Thus, both of the two results given by $M = 2$ and 3 are wrong: one gives the so-called CC from the result based on $M = 2$, and the other a wrong prediction from the result based on $M = 3$. In this way, both of CC and CUP can be avoided by using reliable solutions with a large enough critical predictable time T_c. Similarly, the so-called CP can be avoided by means of the concept of the critical predictable time.

The above examples illustrate the importance and need of introducing the concept of the critical predictable time T_c. In this way, CC, CP and CUP of non-linear dynamic systems can be avoided completely by using reliable solutions with a large enough critical predictable time T_c.

Theoretically speaking, in case of $\sigma = 10, R = 28, b = -8/3$ with the initial condition $x(0) = -15.8, y(0) = -17.48, z(0) = 35.64$, given a 'finite' value of the critical predictable time T_c, one can always determine the order $M \approx T_c/3$ of the truncated Taylor series scheme (3) by means of (5) and the number $K \approx 0.4\,T_c$ of the accurate decimal places of data by means of (6), respectively, although the needed CPU times might be rather long. So, from mathematical review-points, by means of the concept of the critical predictable time T_c, the CU of long-term prediction of chaotic dynamic systems could be avoided, as long

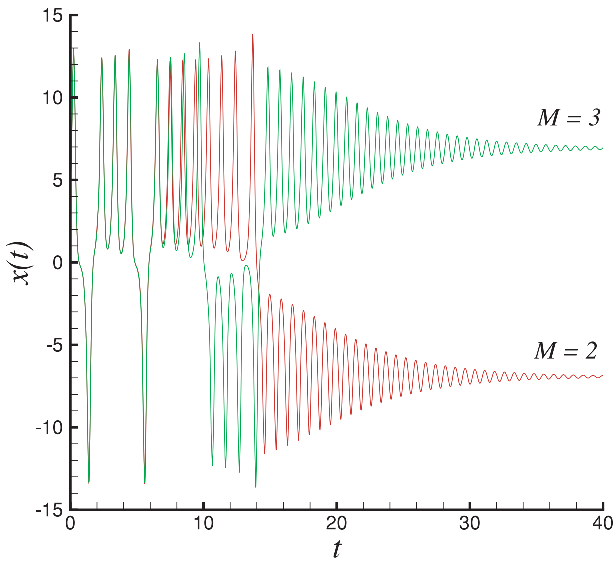

Fig. 8. Comparison of $x(t)$ in case of $\sigma = 10$, $R = 19$, $b = -8/3$ with the initial state $x = y = z = 5$ by means of $\tau = 0.01$ LTU and the Mth-order scheme (3) based on truncated Taylor series. Red line: $M = 2$; green line: $M = 3$.

as we have fast enough computer with large enough memory (RAM). Even so, it is always impossible to get a reliable chaotic solution in an infinite time interval $0 < t < +\infty$, as mentioned in Section 3.

5. On the origin of prediction uncertainty of chaos

Unfortunately, most non-linear dynamic systems describe physical phenomena in nature. Thus, results given by these models should have physical meanings. So, it is necessary to investigate the prediction uncertainty of chaos from physical view-points.

The famous Lorenz eq. (1) is a macroscopical model for climate prediction on earth: it models a unsteady flow occurring in a layer of fluid of uniform depth H with a constant temperature difference ΔT between the upper and lower surfaces, and $x(t)$ is proportional to the intensity of convective motion (Lorenz, 1963). So, it is reasonable that the influence of physical factors in the level of atom and molecule on the climate is completely neglected in Lorenz equation. On one hand, for purpose of climate prediction, measured data are 'unnecessary' to be precise in the level of atom and molecule. On the other hand, as output of a macroscopical model, computational results given by Lorenz equation are 'impossible' to be precise in the microcosmis level.

As mentioned in Section 3, a reliable computational chaotic result with $T_c = 1200$ LTU is obtained by means of $M = 400$ and $K = 800$ with the 'exact' initial condition $x(0) = -15.8$, $y(0) = -17.48$, $z(0) = 35.64$. Note that $K = 800$ corresponds to very precise data. However, according to (5) and (6), it is unnecessary to use so precise data to get a chaotic result with

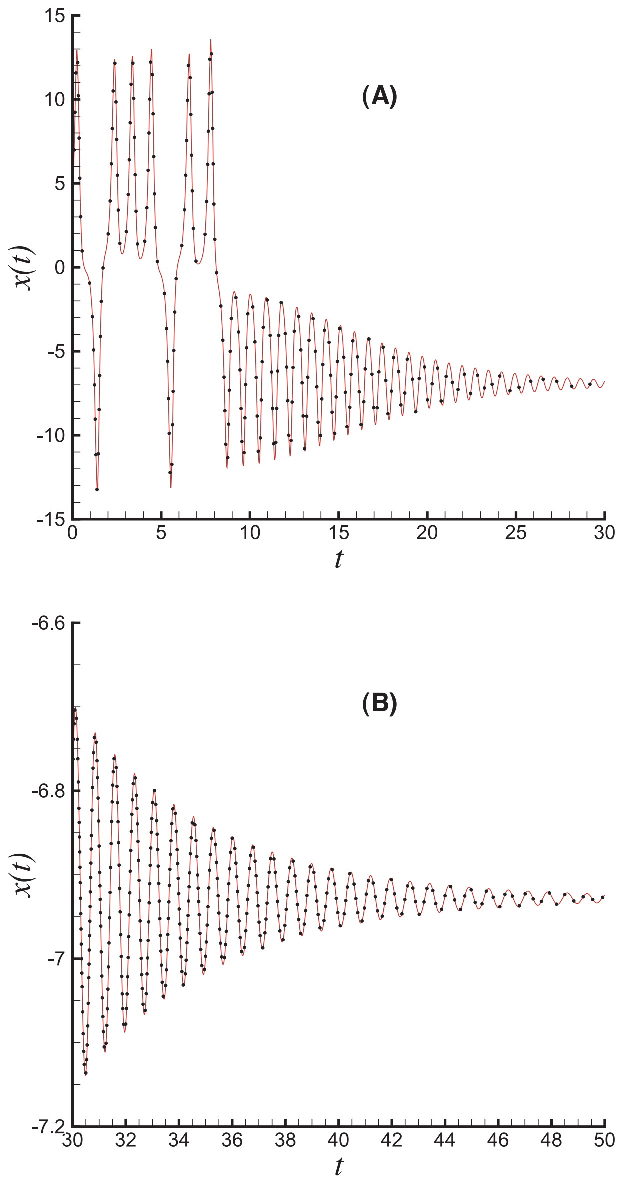

Fig. 9. (a) Comparison of $x(t)$ in case of $\sigma = 10$, $R = 19$, $b = -8/3$ with the initial state $x = y = z = 5$ by means of $\tau = 0.01$ LTU and the Mth-order scheme (3) based on truncated Taylor series in the interval $0 \le t \le 30$. Line: $M = 30$; Symbols: $M = 40$. (b) The same, but in the interval $30 \le t \le 50$.

$T_c = 1200$ LTU. Substituting $T_c = 1200$ LTU into (5) and (6) gives $M = 400$ and $K = 480$, respectively. Thus, mathematically speaking, at least the precise data with 480 decimal places 'must' be used to get a chaotic result reliable in the interval $0 \le t \le 1200$ LTU. This conclusion is obtained with the assumption that the initial condition is exact. Unfortunately, the initial condition is not perfect in practice. It is interesting that substituting $T_c = 1200$ into (7) gives $\Delta x_0 = 10^{-480}$, which agrees well with the previous calculation $K = 480$. This indicates that the initial

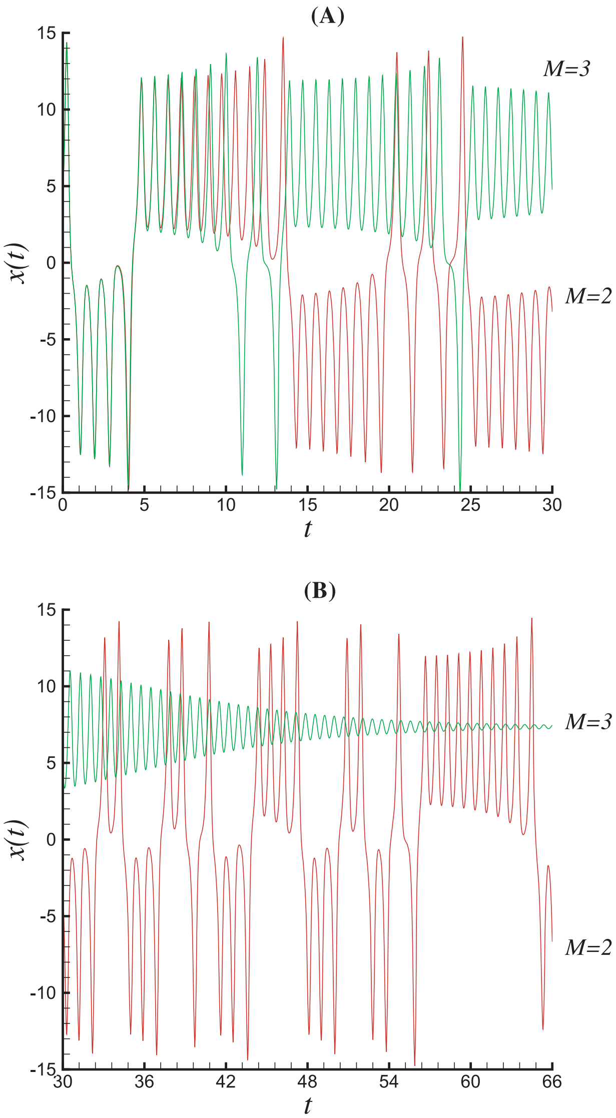

Fig. 10. (a) Comparison of $x(t)$ in case of $\sigma = 10$, $R = 21.5$, $b = -8/3$ with the initial state $x = y = z = 5$ by means of $\tau = 0.01$ LTU and the Mth-order scheme (3) based on truncated Taylor series in the interval $0 \le t \le 30$. Red line: $M = 2$; Green line: $M = 3$. (b) The same, but in the interval $30 \le t \le 66$.

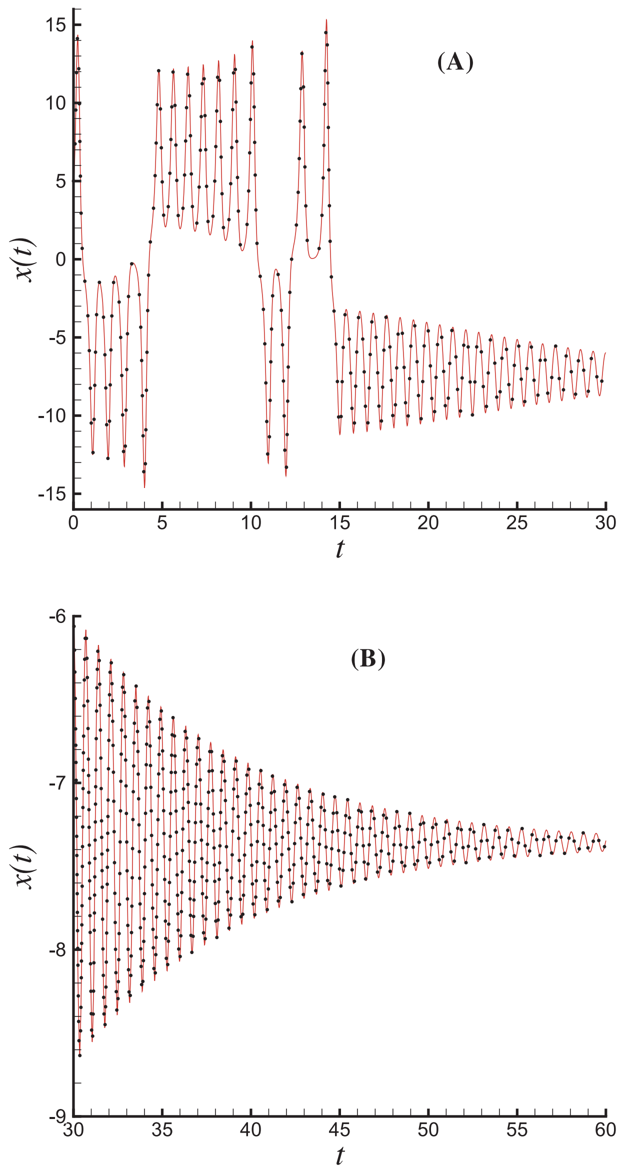

Fig. 11. (a) Comparison of $x(t)$ in case of $\sigma = 10$, $R = 21.5$, $b = -8/3$ with the initial state $x = y = z = 5$ by means of $\tau = 0.01$ LTU and the Mth-order scheme (3) based on truncated Taylor series in the interval $0 \le t \le 30$. Line: $M = 30$; Symbols: $M = 40$. (b) The same, but in the interval $30 \le t \le 60$.

condition must be at least with the same accuracy as all computed data used at each time step. Therefore, from pure mathematical view-points, the initial condition (and all computed data) must be with the precision $\Delta x_0 = 10^{-480}$ to get a reliable chaotic solution with $T_c = 1200$ LTU.

First, to show how small the number 10^{-480} is, let us compare it with some physical constants. According to NASA's Wilkinson Microwave Anisotropy Probe (WMAP) project, the age of the universe is estimated to be about 1.373×10^{10} yr, that is, $T_u \approx 4.33 \times 10^{17}$ s, and its diameter is about 93 billion light years, that is, $d_u \approx 8.8 \times 10^{26}$ m (http://en.wikipedia.org/wiki/Universe). On the other side, helium is the smallest atom with a radius of 32 picometre (http://en.wikipedia.org/wiki/Atom#Size), that is, $r_a \approx 3.2 \times 10^{-11}$ m, and the diameter of the nucleus for a proton in light hydrogen is about 1.6 feotometre (http://en.wikipedia.org/wiki/Atomic_nucleus), that is, $d_n \approx 1.6 \times 10^{-15}$ m. Assume that one 'object' moves a distance of radius of helium or a diameter of a proton in light hydrogen since the beginning of the universe, that is, the Big Bang (http://en.wikipedia.org/wiki/Big_Bang). Then, the corresponding velocities are $u_a = r_a/T_u \approx 7.39 \times 10^{-29}$ (m s^{-1}) and $u_n = d_n/T_u \approx 3.7 \times 10^{-33}$

(m s^{-1}), respectively. However, even dividing them by the speed of light (http://en.wikipedia.org/wiki/Light_speed) $c \approx 3.0 \times 10^9$ (m s^{-1}), which is assumed to be the largest velocity in nature (Einstein, 1905), we have only the dimensionless velocities $\bar{u}_a = r_a/(cT_u) \approx 2.46 \times 10^{-38}$ and $\bar{u}_n = d_n/(cT_u) \approx 1.23 \times 10^{-42}$, respectively. Even so, they are 'much' larger than 10^{-480}, because both $\bar{u}_a/10^{-480} = 2.46 \times 10^{442}$ and $\bar{u}_n/10^{-480} = 1.23 \times 10^{438}$ are 'much' greater even than $d_u/d_n = 5.5 \times 10^{41}$, the ratio of the diameter of the universe to the diameter of the nucleus for a proton in light hydrogen!

Secondly, according to the Heisenberg uncertainty principle in quantum physics (Heisenberg, 1927), the values of certain pairs of conjugate variables (position and momentum, for instance) cannot both be known with arbitrary precision, and any measurement of the position with accuracy $\Delta\delta$ and the momentum with accuracy Δp must satisfy

$$\Delta\delta \ \Delta p \geq \frac{h}{4\pi}, \tag{8}$$

where $h = 6.62606896 \times 10^{-34}$ [J] [S] is Planck's constant (http://en.wikipedia.org/wiki/Planck_constant), with the unit [J] of energy (joule) and the unit [S] of time (s). Rewriting $\Delta p = m \ \Delta v$, where m denotes the mass and v the velocity, one has

$$\Delta v \Delta\delta \geq \frac{h}{4\pi m}. \tag{9}$$

Therefore, the more precisely the velocity is known, the less precisely the position is known. Because Lorenz equation models the flow of fluid on the earth, the worst measurement of a position is with accuracy $\Delta\delta = d_E$, where $d_E = 1.2745 \times 10^7$ (m) is the average diameter of the earth. Then, the most precise measurement of velocity is at most

$$\Delta v \geq \frac{h}{(4\pi d_E)m} = \frac{4.1372 \times 10^{-42}}{m}. \tag{10}$$

Even if m is regarded as the mass of earth, that is, $m = 5.9742 \times 10^{24}$ (kg), the most precise measurement of velocity is at most

$$\Delta v \geq 6.92511 \times 10^{-67} (\mathrm{m\,s^{-1}}). \tag{11}$$

Let $\bar{v}$ denote the dimensionless velocity and U the velocity reference, respectively. The above formula gives

$$\Delta\bar{v} \geq \frac{6.92511 \times 10^{-67}}{U}.$$

According to the special relativity (Einstein, 1905), light propagates fastest in nature. However, even if the velocity of light is used as the reference velocity, that is, $U \approx 3.0 \times 10^9$ (m s^{-1}), the most precise measurement of dimensionless velocity on earth is at most

$$\Delta\bar{v} \geq 2.3 \times 10^{-76}. \tag{12}$$

Therefore, it is impossible to measure a dimensionless velocity more precisely than the above value. Here, it should be emphasized that even the above very tiny number 2.3×10^{-76} is much larger than 10^{-480}: the ratio $(2.3 \times 10^{-76})/10^{-480} = 2.3 \times 10^{404}$ is much greater even than $d_u/d_n = 5.5 \times 10^{41}$, the ratio of the diameter of the universe to the diameter of the nucleus for a proton in light hydrogen! In fact, the ratio $(2.3 \times 10^{-76})/10^{-480}$ is larger than $(d_u/d_n)^7$. Therefore, according to the Heisenberg uncertainty principle in quantum physics, it is 'physically' impossible to give an initial condition with so high precision $\Delta x_0 = 10^{-480}$, which is however 'mathematically' necessary to get a chaotic result reliable in the interval $0 \leq t \leq 1200$ LTU), as mentioned in Section 3.

How can we interpret the above interesting result? It seems unavoidable to use non-linear dynamic models to describe the nature; besides, chaos generally exist in various non-linear dynamic models. However, as mentioned above, to get chaotic results reliable in a long enough time, we need initial condition with precision even higher than the most precise measurement allowed by the Heisenberg uncertainty principle in quantum physics. Note that the precision, which is 'mathematically' necessary for the initial condition and all computed data at different time, is so high that even the quantum fluctuation becomes a very important physical factor and therefore cannot be neglected. Therefore, the famous 'butterfly effect' of Lorenz equation should be replaced by the so-called 'quantum fluctuation effect': even the 'microcosmic' physical uncertainty such as quantum fluctuation may produce a large variations in the long-term macroscopical behaviour of a chaotic dynamic system describing natural phenomena.

Thus, although from mathematical points of view we can indeed obtain reliable chaotic solution with $T_c > 1200$ LTU in case of $\sigma = 10, R = 28, b = -8/3$ with the initial condition $x(0) = -15.8, y(0) = -17.48, z(0) = 35.64$ by means of $\tau = 0.01$ LTU, $M = 400$ and $K = 480$, unfortunately, this 'mathematical' solution with such a high precision has no 'physical' meanings. It should be emphasized that Lorenz equation is a 'macroscopical' model for climate prediction, and thus 'microcosmic' physical factors such as the quantum fluctuation are neglected. However, it is 'mathematically' necessary for Lorenz equation to have the initial condition with such a high precision that the Heisenberg uncertainty principle in quantum physics must be considered 'physically'. This provides us a 'precision paradox of chaos'.

A paradox often brings us much deeper understandings about some thoughts and/or theories. What can such a paradox tell us? It seems that, to avoid this paradox, the following assumptions or view-points should be accepted:

(1) In essence, chaos is physically unpredictable. The origin of the unpredictability of chaos comes essentially from the microcosmic uncertainty, which is described by the Heisenberg uncertainty principle in quantum physics.

(2) Even macroscopical phenomena might be essentially uncertain, and thus, it would be more reliable and more economic to describe them by probability.

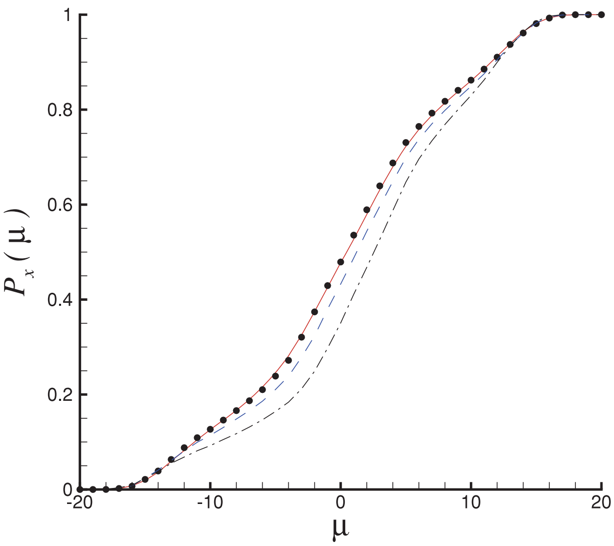

Fig. 12. The probability $P_x(\mu)$ in case of $\sigma = 10$, $R = 28$, $b = -8/3$ and the exact initial condition $x(0) = -15.8$, $y(0) = -17.48$, $z(0) = 35.64$ by means of $\tau = 0.01$ (LTU) with different T_c. Symbols: result given by reliable solution with $T_c = 300$ (LTU); Solid line: result given by reliable solution with $T_c = 75$ (LTU); Dashed line: result given by reliable solution with $T_c = 50$ (LTU); Dash–dotted line: result given by reliable solution with $T_c = 25$ (LTU).

(3) Most of current non-linear dynamic models, which describe complicated macroscopical phenomena such as chaos and turbulence, do not consider the influence of microcosmic uncertainty and, thus, should be modified.

To support the above interpretations for the so-called 'precision paradox of chaos', let us first consider the statistic probability of $x(t)$ less than μ, denoted by $P_x(\mu)$. The probabilities $P_x(\mu)$ obtained by reliable chaotic results with different critical predictable time T_c in case of $\sigma = 10$, $R = 28$, $b = -8/3$ with the exact initial condition $x(0) = -15.8$, $y(0) = -17.48$, $z(0) = 35.64$ by means of $\tau = 0.01$ LTU are as shown in Fig. 12. Note that the probability $P_x(\mu)$ given by $T_c = 75$ LTU agrees well with the probability given by the reliable computational result with $T_c = 300$ LTU. It indicates that one can obtain a 'stable' or 'convergent' probability $P_x(\mu)$ by means of a reliable solution with a proper T_c, which is not necessarily very long. Note that it is much easier to get a reliable chaotic result with $T_c = 75$ LTU than that with $T_c = 1200$ LTU. Therefore, it is much cheaper to get a reliable probability $P_x(\mu)$ than a reliable time-series $x(t)$ with $T_c = 1200$ LTU. So, it seems more reliable and, especially, more 'economic' to describe chaotic phenomena by means of probability. Second, it is well known that chaotic time-series is non-periodic. Assume that a time-series $f(t)$ in the interval $t \in [0, T_\rho]$ has N^+ maximum values at $t = t_i^+$ and N^- minimum values at $t = t_j^-$, respectively, where $i = 0, 1, 2, 3, \ldots, N^+$, $j = 0, 1, 2, 3, \ldots, N^-$ and N^+ and N^- are dependent upon T_ρ. If $f(t)$ is periodic, then $|t_i^+ - t_{i-1}^+|$ and $|t_j^- - t_{j-1}^-|$ are the same for any $1 \le i \le N^+$ and $1 \le j \le N^-$. However, if $f(t)$ is chaotic, then $|t_n^+ - t_{n-1}^+|$ and $|t_n^- - t_{n-1}^-|$ are dependent upon n. Let us define the 'generalized-period'

$$\hat{T}^+ = \sum_{n=1}^{N^+} |t_n^+ - t_{n-1}^+| / N^+$$

and

$$\hat{T}^- = \sum_{n=1}^{N^-} |t_n^- - t_{n-1}^-| / N^-,$$

respectively. Using the reliable result in case of $\sigma = 10$, $R = 28$, $b = -8/3$ with the exact initial condition $x(0) = -15.8$, $y(0) = -17.48$, $z(0) = 35.64$ by means of $M = 400$, $K = 800$ and $\tau = 0.01$ LTU, it is found that the generalized-periods of $x(t), y(t), z(t)$ tend to the stable values $\hat{T}_x^+ = \hat{T}_x^- = 0.96$ LTU, $\hat{T}_y^+ = \hat{T}_y^- = 0.62$ LTU and $\hat{T}_z^+ = \hat{T}_z^- = 0.75$ LTU, respectively, if the used interval is larger than 600 LTU. Such kind of stable values of the generalized-period can be called 'statistic period'. Therefore, from statistic view-point, even a chaotic solution can be regarded as a periodic one in a more general meaning! Furthermore, it is found that the generalized-periods $\hat{T}_x$, $\hat{T}_y$ and $\hat{T}_z$ of Lorenz equation (1) are dependent upon the physical parameters σ, R and b. Our calculations indicate that the generalized-periods $\hat{T}_x$, $\hat{T}_y$ and $\hat{T}_z$ of Lorenz eq. (1) are the same for a non-chaotic solution but are different for a chaotic ones. All of these partly support our above interprets about the so-called 'precision paradox of chaos'.

Thus, although the CU of chaos can be avoided from the mathematical point of view, it is unavoidable from the physical point of view: the so-called 'precision paradox of chaos' suggests us that the origin of the uncertainty of chaos comes from the Heisenberg uncertainty principle in quantum physics and, thus, is not avoidable, forever!

6. Discussions

In this paper a new concept, namely the 'critical predictable time' T_c, is introduced to give a more precise description of computed chaotic solutions of non-linear differential equations: computed chaotic solutions are regarded to be reliable only when $0 < t \le T_c$. This provides us a method or strategy to detect the reliable result from a given computed solution. Besides, it provides us a timescale for the so-called 'long-term': t is regarded to be long-term as long as $t > T_c$. It is also suggested that numerical results beyond the critical predictable time T_c are unpredictable, and thus, all related conclusions based on computed chaotic results beyond the critical predictable time T_c are doubtable. In this way, the numerical phenomena such as CC, CP and computational prediction uncertainty, which are mainly based on long-term properties of computed results, can be avoided, as shown in Section 4. By means of this concept, the famous conclusion 'accurate long-term prediction of chaos is impossible' should be replaced by a more precise conclusion that 'accurate prediction of chaos beyond the critical predictable time T_c is impossible'.

For a non-linear dynamic system with chaotic behaviour, one had to solve it by at least two different computation schemes to get the critical predictable time T_c. Certainly, it is better to use more different computation schemes to investigate the reliability of computed results with chaos: the more, the better. So, the reliability of chaotic solutions is a relative concept: it is dependent on not only non-linear differential equations but also the accuracy of the initial condition, the time step and computation schemes. Without knowing the exact solution, such reliable solutions within the critical predictable time T_c might be the best in practice: they are at least predictable, that is, different computation schemes lead to very close results. Note that even the definition (4) of the critical predictable time is dependent upon the two constants δ and ϵ. Fortunately, the same qualitative conclusions are obtained even by different (but reasonable) values of δ and ϵ. So, all of our conclusions mentioned in this paper have general meanings.

On one hand, the so-called critical predictable time T_c provides us a scale to investigate chaos more precisely. On the other hand, the symbolic computation software (such as MATHEMATICA) provide us a convenient way to investigate the influence of the truncation error, the round-off error, and the inaccuracy of the initial condition on the critical predictable time T_c. It is found that T_c is directly proportional to M, the order of the truncated Taylor series scheme (3), and K, the number of decimal places of all computed data. Besides, the precision of the initial condition must increase exponentially as T_c enlarges. For example, in case of $\sigma = 10, R = 28, b = -8/3$ with the initial condition $x(0) = -15.8, y(0) = -17.48, z(0) = 35.64$, we obtain a reliable chaotic result with $T_c = 1200$ LTU by means of $\tau = 0.01$ LTU, $M = 400$ and $K = 800$. Such a reliable chaotic solution in so long time is reported for the first time. Theoretically speaking, given a finite value of the critical predictable time T_c, we can always get a reliable chaotic result in the interval $0 \le t \le T_c$ by means of a high-performance computer with large enough memory (RAM) and fast enough CPU, although the needed CPU time might be rather long. Therefore, in essence, the CUP for chaos can be avoided, if only from the mathematical points of view.

However, the precision of the initial condition and the computed data at each time step needed for a large T_c (such as $T_c =$ 1200 LTU) is 'mathematically' so high that such precise data is 'physically' impossible due to Heisenberg uncertainty principle in quantum physics. Note that the precision, which is 'mathematically' necessary for the initial condition and all computed data at different time, is so high that even the quantum fluctuation becomes a very important physical factor and therefore cannot be neglected. However, as a macroscopical model for climate prediction on Earth, Lorenz equation completely neglects the influence of physical factors in the level of atom and molecule on the climate. This provides us the so-called 'precision paradox of chaos', which implies that the prediction uncertainty of chaos is physically unavoidable; besides, even the macroscopical phenomena might be essentially stochastic and thus should be described by probability more economically.

Many non-linear evaluation equations for macroscopical phenomena, such as Lorenz equation for climate prediction and Navier–Stokes equation for turbulent viscous flows, completely neglect the influence of physical factors in the level of atom and molecule. However, the so-called 'precision paradox of chaos' suggests that this might be wrong: even the microcosmic physical factors should be considered for non-linear dynamic systems that model complicated macroscopical phenomena. It is well known that turbulent flows are much more complicated than chaos. Our work mentioned in this paper suggests that one should be very careful in applying numerical schemes to investigate turbulent flows. Today, the direct numerical simulation (DNS) is frequently used in the computational fluid dynamics (CFD) to simulate turbulence flows (e.g. Le et al., 1997; Moin and Mahesh, 1998; Moser et al., 1999; Scardovelli and Zaleski, 1999; Martin et al., 2006). However, it is a pity that the sensitivity of the DNS results to the inaccuracy of the initial condition, the round-off error and the truncation error has not been studied systematically, mainly because the DNS is rather time-consuming. Without a method or strategy to detect the reliability of a given DNS result, we have many reasons to assume that something without physical meanings (similar to CP and CC) might be contained in the so-called DNS 'solutions' for turbulence, and thus, conclusions based on such kind of unreliable computed results might be doubtable. More importantly, all models for turbulent flows completely neglect the influence of microcosmic physical factors. Also, this might be one of the reasons why there is no satisfactory model to describe all turbulent flows precisely.

Note that the concept of the critical predictable time T_c is not new: it is rather similar to the so-called 'decoupling time' mentioned by Teixeira et al. (2007). However, this concept has never been obtained enough recognition. In this paper, we show the importance of such concept for the reliability of computed results and also for the avoidance of computational prediction uncertainty, CC and CP. More importantly, the concept of the critical predictable time T_c greatly deepens and enriches our understanding about chaos, not only mathematically but also physically.

Non-linear dynamic systems describing chaos or turbulence might be much more complicated than we thought: we should feel awe to them. It is the time for us to consider seriously the reliability of a mass of computed chaotic or turbulent results reported every day.

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References

Ashwin, P. 2003. Synchronization from chaos. *Nature* **42**, 384–385.

Cloutman, L. D. 1996. A note on the stability and accuracy of finite difference approximations to differential equations. *Report No. UCRL-ID-125549*, Lawrence Livermore National Laboratory, Livermore, CA.

Cloutman, L. D. 1998. Chaos and instabilities in finite difference approximations to nonlinear differential equations. *Report No. UCRL-ID-131333*, Lawrence Livermore National Laboratory, Livermore, CA.

Einstein, E. 1905. On the electrodynamics of moving bodies. *Annalen der Physik* **17**, 891–921.

Hanski, I., Turchin, P., Korplmäki, E. and Henttonen, H. 1993. Population oscillations of boreal rodents: regulation by mustelid predators leads to chaos. *Nature* **364**, 232–235.

Heisenberg, W. 1927. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift für Physik* **43**, 172–198.

Le, H., Moin, P. and Kim, J. 1997. Direct numerical simulation of turbulent flow over a backward-facing step, *J. Fluid Mech.* **330**, 349–374.

Li, T. Y. and Yorke, J. A. 1975. Period three implies chaos. *Am. Math. Mon.* **10**, 985–992.

Lorenz, E. N. 1963. Deterministic nonperiodic flow. *J. Atmos. Sci.* **20**, 130–141.

Lorenz, E. N. 1989. Computational chaos: a prelude to computational instability. *Physica D* **35**, 299–317.

Lorenz, E. N. 2006. Computational periodicity as observed in a simple system. *Tellus* **58A**, 549–557.

Malescio, G. 2005. Predicting with unpredictability. *Nature* **434**, 1073.

Martin, M. P., Taylor, E. M., Wu, M. and Weirs, V. G. 2006. A bandwidth-optimized WENO scheme for the effective direct numerical simulation of compressible turbulence. *J. Comp. Phys.* **220**, 270–289.

Moin, P. and Mahesh, K. 1998. Direct numerical simulation: a tool in turbulence research. *Ann. Rev. Fluid Mech.* **30**, 539–578.

Moser, R. D., Kim, J. and Mansour, N. N. 1999. Direct numerical simulation of turbulent channel flow up to Re = 590. *Phys. Fluids* **11**, 943–945.

Parker, T. S. and Chua, L. O. 1989. *Practical Numerical Algorithms for Chaotic Systems*. Springer-Verlag, New York.

Scardovelli, R. and Zaleski, S. 1999. Direct numerical simulation of free-surface and interfacial flow. *Ann. Rev. Fluid Mech.* **31**, 567–603.

Stewart, I. 2000. The Lorenz attractor exists. *Nature* **406**, 948–949.

Teixeira, J., Reynolds, C. A. and Judd, K. 2007. Time step sensitivity of nonlinear atmospheric models: numerical convergence, truncation error growth, and ensemble design. *J. Atmos. Sci.* **64**, 175–189.

Teixeira, J., Reynolds, C. A. and Judd, K. 2008. Reply to Yao and Hughes' comments. *J. Atmos. Sci.* **65**, 683–684.

Yao, L. S. and Hughes, D. 2008a. Comment on 'Computational periodicity as observed in a simple system' By Edward N. Lorenz (2006). *Tellus* **60A**, 803–805.

Yao, L. S. and Hughes, D. 2008b. Comments on 'Time step sensitivity of nonlinear atmospheric models: numerical convergence, truncation error growth, and ensemble design'. *J. Atmos. Sci.* **65**, 681–682.

Supporting information

Additional supporting information may be found in the online version of this article.

Appendix S1 Numerical results of Lorenz equation with $T_c = 1200$ LTU.

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