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# On the relationship between the homotopy analysis method and Euler transform

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ABSTRACT

A new transform, namely the homotopy transform, is defined for the first time. Then, it is proved that the famous Euler transform is only a special case of the so-called homotopy transform which depends upon one non-zero auxiliary parameter $\hbar$ and two convergent series $\sum_{k=1}^{+\infty}\alpha_{1,k} = 1$ and $\sum_{k=1}^{+\infty}\beta_{1,k} = 1$. In the frame of the homotopy analysis method, a general analytic approach for highly nonlinear differential equations, the so-called homotopy transform is obtained by means of a simple example. This fact indicates that the famous Euler transform is equivalent to the homotopy analysis method in some special cases. On one side, this explains why the convergence of the series solution given by the homotopy analysis method can be guaranteed. On the other side, it also shows that the homotopy analysis method is more general and thus more powerful than the Euler transform.



## 1. Introduction

Solving nonlinear problems is inherently difficult even by means of numerical methods, and the stronger the nonlinearity, the more intractable solutions becomes. Very a few nonlinear problems have closed-form solutions. In most cases, approximate techniques are used to give asymptotic results or series solution of a given nonlinear equation. Perturbation techniques [1–6] are widely applied to solve nonlinear equations in science and engineering. However, it is a pity that perturbation techniques are in principle based on small/large physical parameters (perturbation quantity) and thus perturbation approximations often break down as the nonlinearity becomes strong. To avoid this restrictions, some non-perturbative techniques, such as Lyapunov artificial small parameter method [7], Adomian decomposition method [8,9] and the $\delta$-expansion method [10,11], are developed. Although these non-perturbative techniques seem to have nothing to do with small/large physical parameters, they however cannot guarantee the convergence of solution series, and thus are still valid only for problems with weak nonlinearity, too.

In 1992, Liao [12] first introduced the homotopy [13], a basic concept in topology, to propose an analytic technique for strongly nonlinear problems, namely the homotopy analysis method (HAM). Thereafter, the HAM has been improved step by step [14–20] and has been widely applied in science, engineering and finance [21–29]. Different from perturbation techniques, the HAM is independent of any small/large physical parameters. Besides, different from perturbation and traditional non-perturbative techniques, the HAM provides us a simple way to ensure the convergence of solution series. Therefore, the

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HAM is valid even for problems with strong nonlinearity. Especially, by means of the HAM, some new solutions [30,31] of a few nonlinear differential equations have been found, which were neglected by other analytic approximation methods and even by numerical techniques. This shows the great potential of the HAM.

Euler transform [32] is widely applied to accelerate a convergent series or sometimes even to make a divergent series convergent. In this paper, the relationship between the homotopy analysis method and the Euler transform is investigated. In Section 2, the so-called generalized Taylor series is derived, which leads to the so-called homotopy transform in Section 3. Then, it is proved that the Euler transform is only a special case of the homotopy transform. In Section 4, we show by a simple example that the homotopy transform can be derived in the frame of the HAM in some special cases. Thus, the Euler transform is equivalent to the HAM, but only in some special cases. On one side, this explains why the HAM can ensure the convergence of solution series. On the other side, it also indicates that the HAM is more general and thus should be more powerful than the Euler transform.

## 2. Generalized Taylor series

**Definition 1.** Let $p$ be a complex number. A complex function $A(p)$ is called a deformation function if it satisfies

$$A(0)=0, \quad A(1)=1 \tag{1}$$

and is analytic in the region $|p|\leqslant 1$ so that its Maclaurin series $\sum_{k=1}^{+\infty}\alpha_{1,k}p^k$ is convergent in the region $|p|\leqslant 1$, say,

$$A(p)=\sum_{k=1}^{+\infty}\alpha_{1,k}\,p^k, \quad |p|\leqslant 1, \tag{2}$$

holds.

**Theorem 1.** *Let $p, z, z_0$ and $\hbar\neq 0$ be complex numbers, $A(p)$ and $B(p)$ two deformation functions satisfying $A(0)=B(0)=0$, $A(1)=B(1)=1$, whose Maclaurin series $\sum_{k=1}^{+\infty}\alpha_{1,k}\,p^k$ and $\sum_{k=1}^{+\infty}\beta_{1,k}\,p^k$ are absolutely convergent in the region $|p|\leqslant 1$. Suppose $|1+\hbar|<1$ and $|B(p)|\leqslant 1$ for $p\in[0,1]$. Define*

$$\vec{\alpha}=\{\alpha_{1,1},\alpha_{1,2},\alpha_{1,3},\ldots\}, \quad \vec{\beta}=\{\beta_{1,1},\beta_{1,2},\beta_{1,3},\ldots\}, \tag{3}$$

$$\alpha_{m,k}=\sum_{i=m-1}^{k-1}\alpha_{m-1,i}\alpha_{1,k-i} \quad (m\geqslant 2,\ k\geqslant m), \tag{4}$$

$$\beta_{0,0}=1,\ \beta_{0,k}=0 \quad (k\geqslant 1), \quad \beta_{m,k}=\sum_{i=m-1}^{k-1}\beta_{m-1,i}\beta_{1,k-i} \quad (m\geqslant 2,\ k\geqslant m), \tag{5}$$

and

$$T_{m,k}(\hbar,\vec{\alpha},\vec{\beta})=(-\hbar)^k\sum_{n=0}^{m-k}\sum_{r=0}^{n}\binom{k+r-1}{r}(1+\hbar)^r\sum_{s=0}^{n-r}\alpha_{k,k+s}\beta_{r,n-s} \tag{6}$$

for $m\geqslant 1$ and $1\leqslant k\leqslant m$. If a complex function $f(z)$ is analytic at $z_0$ but singular at $\xi_k(k=1,2,\ldots,M_0)$, where $M_0$ may be infinity, the series

$$f(z_0)+\sum_{k=1}^{+\infty}\left[\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k\right]T_{m,k}(\hbar,\vec{\alpha},\vec{\beta}), \tag{7}$$

converges to $f(z)$ in the region $D=\bigcap_{k=0}^{M}S_k$, where $S_k=\{z:|\omega_k|>1\}$ and the complex numbers $\omega_k(k=0,1,2,3,\ldots,M,\ M$ may be infinity) are the solutions of the algebraic equation

$$B(\omega_0)=(1+\hbar)^{-1},$$

or

$$1-(1+\hbar)B(\omega_k)+\hbar\left(\frac{z-z_0}{\xi_n-z_0}\right)A(\omega_k)=0,\ 1\leqslant n\leqslant M_0.$$

**Proof:.** Using the definition (1) of the deformation function, we have

$$A(0)=B(0)=0, \quad A(1)=\sum_{k=1}^{+\infty}\alpha_{1,k}=1,\ B(1)=\sum_{k=1}^{+\infty}\beta_{1,k}=1. \tag{8}$$

Write

$$[A(p)]^m = \left(\sum_{k=1}^{+\infty} \alpha_{1,k} p^k\right)^m = \sum_{k=m}^{+\infty} \alpha_{m,k}\, p^k, \quad m \geqslant 1, \tag{9}$$

$$[B(p)]^m = \left(\sum_{k=1}^{+\infty} \beta_{1,k} p^k\right)^m = \sum_{k=m}^{+\infty} \beta_{m,k}\, p^k, \quad m \geqslant 0, \tag{10}$$

If $\alpha_{m-1,k}$ $(m \geqslant 2, k \geqslant m-1)$ are known, it follows that

$$\begin{aligned}[A(p)]^m &= \sum_{k=m}^{+\infty} \alpha_{m,k} p^k = \left(\sum_{k=1}^{+\infty} \alpha_{1,k}\, p^k\right)^{m-1} \left(\sum_{k=1}^{+\infty} \alpha_{1,k}\, p^k\right) \\ &= \left(\sum_{i=m-1}^{+\infty} \alpha_{m-1,i}\, p^i\right)\left(\sum_{j=1}^{+\infty} \alpha_{1,j}\, p^j\right) = \sum_{k=m}^{+\infty} p^k \left(\sum_{i=m-1}^{k-1} \alpha_{m-1,i}\, \alpha_{1,k-i}\right)\end{aligned}$$

which gives a recurrence formula

$$\alpha_{m,k} = \sum_{i=m-1}^{k-1} \alpha_{m-1,i}\, \alpha_{1,k-i}, \quad m \geqslant 2,\ k \geqslant m. \tag{11}$$

Similarly, it holds

$$\beta_{m,k} = \sum_{i=m-1}^{k-1} \beta_{m-1,i}\, \beta_{1,k-i}, \quad m \geqslant 2,\ k \geqslant m. \tag{12}$$

Besides, according to the definition (10), it holds

$$\beta_{0,0} = 1, \quad \beta_{0,k} = 0 \ \ (k \geqslant 1). \tag{13}$$

Let us define the two complex variables

$$\gamma_k = \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k, \quad k \geqslant 1, \tag{14}$$

$$\tau = z_0 - \frac{\hbar(z-z_0)A(p)}{1-(1+\hbar)B(p)}, \quad \hbar \neq 0, \tag{15}$$

and construct such a related complex function

$$F(p) = f(\tau) = f\left(z_0 - \frac{\hbar(z-z_0)A(p)}{1-(1+\hbar)B(p)}\right). \tag{16}$$

According to (8), it holds $\tau = z_0$ when $p = 0$ and $\tau = z$ when $p = 1$, respectively. Therefore, we have

$$F(0) = f(z_0), \quad F(1) = f(z). \tag{17}$$

In other words, $F(p)$ is a homotopy, i.e. $F(p) : f(z_0) \sim f(z)$. Writing

$$\delta\tau = -\frac{\hbar(z-z_0)A(p)}{1-(1+\hbar)B(p)}, \tag{18}$$

we have $F(p) = f(z_0 + \delta\tau)$. Provided that $|\delta\tau|$ is sufficiently small and $|(1+\hbar)B(p)| < 1$ holds, then, by means of the definitions (11) and (12), the Maclaurin series of $F(p)$ is

$$\begin{aligned}&f(z_0) + \sum_{k=1}^{+\infty} \frac{f^{(k)}(z_0)}{k!}(\delta\tau)^k \\ &= f(z_0) + \sum_{k=1}^{+\infty} \frac{f^{(k)}(z_0)}{k!}\left[-\frac{\hbar(z-z_0)A(p)}{1-(1+\hbar)B(p)}\right]^k \\ &= f(z_0) + \sum_{k=1}^{+\infty} \gamma_k(-\hbar)^k A^k(p)[1-(1+\hbar)B(p)]^{-k} \\ &= f(z_0) + \sum_{k=1}^{+\infty} \gamma_k(-\hbar)^k A^k(p) \sum_{r=0}^{+\infty} \binom{k+r-1}{r}(1+\hbar)^r B^r(p)\end{aligned}$$

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$$
\begin{aligned}
&= f(z_0) + \sum_{k=1}^{+\infty}\sum_{r=0}^{+\infty}\gamma_k\binom{k+r-1}{r}(-\hbar)^k(1+\hbar)^r\sum_{i=k}^{+\infty}\alpha_{k,i}p^i\sum_{j=r}^{+\infty}\beta_{r,j}\,p^j\\
&= f(z_0) + \sum_{k=1}^{+\infty}\sum_{r=0}^{+\infty}\gamma_k\binom{k+r-1}{r}(-\hbar)^k(1+\hbar)^r\sum_{s=k+r}^{+\infty}p^s\left(\sum_{i=k}^{s-r}\alpha_{k,i}\,\beta_{r,s-i}\right)\\
&= f(z_0) + \sum_{s=1}^{+\infty}p^s\sum_{k=1}^{s}\gamma_k(-\hbar)^k\sum_{r=0}^{s-k}\binom{k+r-1}{r}(1+\hbar)^r\left(\sum_{i=k}^{s-r}\alpha_{k,i}\,\beta_{r,s-i}\right)\\
&= f(z_0) + \sum_{n=1}^{+\infty}\sigma_n\,p^n,
\end{aligned}
\tag{19}
$$

where

$$
\sigma_n = \sum_{k=1}^{n}\gamma_k(-\hbar)^k\sum_{r=0}^{n-k}\binom{k+r-1}{r}(1+\hbar)^r\left(\sum_{i=k}^{n-r}\alpha_{k,i}\,\beta_{r,n-i}\right). \tag{20}
$$

Let $\omega_k(k=0,1,2,3,\ldots,M,\ M$ may be infinity) denote all singularities of $F(p)$. Due to (16), $\omega_0$, the solution of the equation

$$
1-(1+\hbar)B(\omega_0)=0, \tag{21}
$$

is obviously a singularity of $F(p)$. Besides, each original singularity $\xi_k$ $(1\leqslant k\leqslant M_0)$ of $f(z)$ gives corresponding singularity (or singularities) $\omega_n$ governed by the equation

$$
z_0 - \frac{\hbar(z-z_0)A(\omega_n)}{1-(1+\hbar)B(\omega_n)} = \xi_k, \tag{22}
$$

so that $\omega_n$ $(1\leqslant n\leqslant M)$ must be the solution(s) of the following equations

$$
1-(1+\hbar)B(\omega_n)+\hbar\left(\frac{z-z_0}{\xi_k-z_0}\right)A(\omega_n)=0,\quad 1\leqslant k\leqslant M_0. \tag{23}
$$

The Maclaurin series (19) converges to $F(1)=f(z)$ at $p=1$ if and only if all singularities $\omega_k$ of $F(p)$ are out of the region $|p|\leqslant 1$, say,

$$
|\omega_k|>1,\quad k=0,1,2,\ldots,M. \tag{24}
$$

Thus, by means of (19) and (20), it holds

$$
\begin{aligned}
f(z) = F(1) &= f(z_0) + \lim_{m\to+\infty}\sum_{n=1}^{m}\sigma_n\\
&= f(z_0) + \lim_{m\to+\infty}\sum_{n=1}^{m}\sum_{k=1}^{n}\gamma_k(-\hbar)^k\sum_{r=0}^{n-k}\binom{k+r-1}{r}(1+\hbar)^r\left(\sum_{i=k}^{n-r}\alpha_{k,i}\,\beta_{r,n-i}\right)\\
&= f(z_0) + \lim_{m\to+\infty}\sum_{k=1}^{m}\gamma_k(-\hbar)^k\sum_{n=k}^{m}\sum_{r=0}^{n-k}\binom{k+r-1}{r}(1+\hbar)^r\left(\sum_{i=k}^{n-r}\alpha_{k,i}\,\beta_{r,n-i}\right)\\
&= f(z_0) + \lim_{m\to+\infty}\sum_{k=1}^{m}\left[\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k\right]T_{m,k}(\hbar,\vec{\alpha},\vec{\beta})
\end{aligned}
\tag{25}
$$

in the region $D=\bigcap_{k=0}^{M}S_k$, where $S_k=\{z:|\omega_k|>1\}$ and

$$
\begin{aligned}
T_{m,k}(\hbar,\vec{\alpha},\vec{\beta}) &= (-\hbar)^k\sum_{n=k}^{m}\sum_{r=0}^{n-k}\binom{k+r-1}{r}(1+\hbar)^r\left(\sum_{i=k}^{n-r}\alpha_{k,i}\,\beta_{r,n-i}\right)\\
&= (-\hbar)^k\sum_{n=0}^{m-k}\sum_{r=0}^{n}\binom{k+r-1}{r}(1+\hbar)^r\left(\sum_{i=k}^{n+k-r}\alpha_{k,i}\,\beta_{r,n+k-i}\right)\\
&= (-\hbar)^k\sum_{n=0}^{m-k}\sum_{r=0}^{n}\binom{k+r-1}{r}(1+\hbar)^r\left(\sum_{s=0}^{n-r}\alpha_{k,k+s}\,\beta_{r,n-s}\right). \quad \square
\end{aligned}
\tag{26}
$$

**Definition 2.** The series (7) is called the generalized Taylor series of the complex function $f(z)$ analytic at $z=z_0$.

## 3. Homotopy transform

Write the sequence $s_n = \sum_{k=0}^{n} u_k$ for a series $\sum_{k=0}^{+\infty} u_k$. Due to Agnew's [32] definition, the Euler transform, denoted by $\mathscr{E}(q)$, of the sequence $\{s_n\}$ is the sequence $\{w_n\}$ defined by

$$w_n = \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} s_k. \tag{27}$$

Note that the so-called generalized Taylor series (7) is valid for a complex function and is dependent upon one auxiliary parameter $\hbar$ and the two complex analytic functions $A(p)$ and $B(p)$ with their Maclaurin series

$$A(p) = \sum_{k=1}^{+\infty} \alpha_{1,k}\, p^k, \quad B(p) = \sum_{k=1}^{+\infty} \beta_{1,k}\, p^k$$

under the restriction

$$\sum_{k=1}^{+\infty} \alpha_{1,k} = 1, \quad \sum_{k=1}^{+\infty} \beta_{1,k} = 1.$$

Here, the two convergent series $\sum_{k=1}^{+\infty} \alpha_{1,k}$ and $\sum_{k=1}^{+\infty} \beta_{1,k}$ are derived from the two analytic functions $A(p)$ and $B(p)$, the so-called deformation functions. However, the definition (6) can be generalized by directly defining the two convergent series $\sum_{k=1}^{+\infty} \alpha_{1,k} = 1$ and $\sum_{k=1}^{+\infty} \beta_{1,k} = 1$. For example,

$$\alpha_{1,k} = (1-\gamma)\gamma^{k-1}, \quad |\gamma| < 1 \tag{28}$$

$$\beta_{1,k} = \frac{6}{(k\,\pi)^2}, \tag{29}$$

and so on. There are many such kinds of convergent series. Thus, we can define a new transform for a sequence as below.

**Definition 3.** Given one non-zero auxiliary parameter $\hbar$ and two convergent series $\sum_{k=1}^{+\infty} \alpha_{1,k} = 1$ and $\sum_{k=1}^{+\infty} \beta_{1,k} = 1$, the so-called homotopy transform, denoted by $\mathscr{T}(\hbar, \vec{\alpha}, \vec{\beta})$, of a series $\sum_{k=0}^{+\infty} u_k$, is a sequence $\{\mu_n\}$ defined by

$$\mu_n = u_0 + \sum_{k=1}^{n} u_k\, T_{n,k}(\hbar, \vec{\alpha}, \vec{\beta}), \tag{30}$$

where $T_{n,k}(\hbar, \vec{\alpha}, \vec{\beta})$ is given by (6) under the definitions (3)–(5). The series $\sum_{k=0}^{+\infty} u_k$ is called summable by the homotopy transform $\mathscr{T}(\hbar, \vec{\alpha}, \vec{\beta})$ if $\mu_n$ tends to a bounded value as $n \to +\infty$.

**Lemma 1.** *If $\alpha_{1,1} = \beta_{1,1} = 1$, and $\alpha_{1,k} = \beta_{1,k} = 0$ for $k > 1$, corresponding to $A(p) = B(p) = p$, then*

$$T_{m,k}(\hbar, \vec{\alpha}, \vec{\beta}) = \Phi_{m,k}(\hbar), \quad k \geqslant m, \tag{31}$$

*where $T_{m,k}(\hbar, \vec{\alpha}, \vec{\beta})$ is defined by (6) and $\Phi_{m,k}(\hbar)$ is given by*

$$\Phi_{m,k}(\hbar) = (-\hbar)^k \sum_{n=0}^{m-k} \binom{n+k-1}{n} (1+\hbar)^n, \quad m \geqslant 1,\ 1 \leqslant k \leqslant m. \tag{32}$$

**Proof:.** When $\alpha_{1,1} = \beta_{1,1} = 1$ and $\alpha_{1,k} = \beta_{1,k} = 0$ for $k > 1$, according to the definitions (3)–(5), we have

$$\alpha_{i,j} = \beta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & j > i, \end{cases}$$

for $i \geqslant 0$, $j \geqslant i$. Then, according to the definitions (6) and (32), it holds for $m \geqslant 1,\ 1 \leqslant k \leqslant m$ that

$$\begin{aligned} T_{m,k}(\hbar, \vec{\alpha}, \vec{\beta}) &= (-\hbar)^k \sum_{n=0}^{m-k} \sum_{r=0}^{n} \binom{k+r-1}{r} (1+\hbar)^r \alpha_{k,k}\, \beta_{r,n} \\ &= (-\hbar)^k \sum_{n=0}^{m-k} \binom{k+n-1}{n} (1+\hbar)^r \alpha_{k,k}\, \beta_{n,n} \\ &= (-\hbar)^k \sum_{n=0}^{m-k} \binom{k+n-1}{n} (1+\hbar)^r \\ &= \Phi_{m,k}(\hbar). \quad \square \end{aligned}$$

**Theorem 2.** *Write the sequence $s_n = \sum_{k=0}^{n} u_k$ for a series $\sum_{k=0}^{+\infty} u_k$ and let $q$ and $\hbar$ be complex numbers. For given two convergent series $\sum_{k=1}^{+\infty}\alpha_{1,k} = 1$ and $\sum_{k=1}^{+\infty}\beta_{1,k} = 1$, the Euler transform $\mathscr{E}(q)$ of the sequence $\{s_n\}$ is the same as the homotopy transform $\mathscr{T}(\hbar, \vec{\alpha}, \vec{\beta})$ of the series $\sum_{k=0}^{+\infty} u_k$ in case of $\hbar = -q$, $\alpha_{1,1} = \beta_{1,1} = 1$ and $\alpha_{1,k} = \beta_{1,k} = 0$ for $k > 1$.*

**Proof:.** Due to Agnew's [32] definition (27) of Euler transform $\mathscr{E}(q)$, the sequence $\{\hat{\mu}_m\}$ given by the Euler transform of the sequence $\{s_n\}$ reads

$$\begin{aligned}\hat{\mu}_m &= \sum_{k=0}^{m}\binom{m}{k} q^k(1-q)^{m-k} s_k \\ &= \sum_{k=0}^{m}\binom{m}{k} q^k(1-q)^{m-k}\sum_{n=0}^{k} u_n = \sum_{n=0}^{m} u_n \sum_{k=n}^{m}\binom{m}{k} q^k(1-q)^{m-k} \\ &= u_0\sum_{k=0}^{m}\binom{m}{k} q^k(1-q)^{m-k} + \sum_{n=1}^{m} u_n\sum_{k=n}^{m}\binom{m}{k} q^k(1-q)^{m-k} \\ &= u_0 + \sum_{n=1}^{m} u_n\sum_{k=n}^{m}\binom{m}{k} q^k(1-q)^{m-k}\end{aligned} \tag{33}$$

According to Lemma 1, it holds $T_{m,n}(\hbar, \vec{\alpha}, \vec{\beta}) = \Phi_{m,n}(\hbar)$ when $\alpha_{1,1} = \beta_{1,1} = 1$ and $\alpha_{1,k} = \beta_{1,k} = 0$ $(k > 1)$. Therefore, by means of the definition (32), the sequence $\{\mu_m\}$ given by the homotopy transform $\mathscr{T}(\hbar, \vec{\alpha}, \vec{\beta})$ in case of $\hbar = -q$, $\alpha_{1,1} = \beta_{1,1} = 1$ and $\alpha_{1,k} = \beta_{1,k} = 0$ $(k > 1)$ is given by

$$\mu_m = u_0 + \sum_{n=1}^{m} u_n \Phi_{m,n}(-q) = u_0 + \sum_{n=1}^{m} u_n q^n \sum_{k=0}^{m-n}\binom{n+k-1}{k}(1-q)^k. \tag{34}$$

Enforcing $\hat{\mu}_m = \mu_m$ and comparing (33) with (34), it remains to show that

$$q^n\sum_{k=0}^{m-n}\binom{n+k-1}{k}(1-q)^k = \sum_{k=n}^{m}\binom{m}{k} q^k(1-q)^{m-k}, \quad 1 \leqslant n \leqslant m. \tag{35}$$

When $1 \leqslant n \leqslant m$, we have

$$\begin{aligned}& q^n\sum_{k=0}^{m-n}\binom{n+k-1}{k}(1-q)^k = q^n\sum_{k=0}^{m-n}\binom{n+k-1}{k}\sum_{r=0}^{k}\binom{k}{r}(-q)^r \\ &= \sum_{r=0}^{m-n}(-q)^{n+r}(-1)^n\sum_{k=r}^{m-n}\binom{n+k-1}{k}\binom{k}{r}\end{aligned} \tag{36}$$

and

$$\begin{aligned}& \sum_{k=n}^{m}\binom{m}{k} q^k(1-q)^{m-k} = \sum_{k=n}^{m}\binom{m}{k} q^k\sum_{r=0}^{m-k}\binom{m-k}{r}(-q)^r \\ &= \sum_{k=n}^{m}\binom{m}{k}(-1)^k\sum_{r=0}^{m-k}\binom{m-k}{r}(-q)^{k+r} \\ &= \sum_{k=0}^{m-n}\binom{m}{k+n}(-1)^{k+n}\sum_{r=0}^{m-n-k}\binom{m-n-k}{r}(-q)^{n+k+r} \\ &= \sum_{s=0}^{m-n}(-q)^{n+s}(-1)^n\sum_{k=0}^{s}(-1)^k\binom{m}{k+n}\binom{m-n-k}{s-k} \\ &= \sum_{r=0}^{m-n}(-q)^{n+r}(-1)^n\sum_{k=0}^{r}(-1)^k\binom{m}{k+n}\binom{m-n-k}{r-k} \\ &= \sum_{r=0}^{m-n}(-q)^{n+r}(-1)^n\sum_{k=0}^{r}(-1)^k\binom{m}{m-n-r}\binom{r+n}{k+n},\end{aligned} \tag{37}$$

where we use such a formula that in the relevant ranges it holds

$$\binom{m}{k+n}\binom{m-k-n}{r-k} = \binom{m}{m-n-r}\binom{r+n}{k+n}. \tag{38}$$

So, by (35)–(37), we need to show

$$\sum_{k=r}^{m-n}\binom{n+k-1}{k}\binom{k}{r}=\sum_{k=0}^{r}(-1)^k\binom{m}{m-n-r}\binom{r+n}{k+n}, 1\leqslant n\leqslant m. \tag{39}$$

Noticing that, for $n\geqslant 1$ and sufficiently small $x$,

$$\sum_{r=0}^{+\infty}x^r\binom{n+r-1}{r}=(1-x)^{-n}, \tag{40}$$

and that

$$\begin{aligned}&\sum_{r=0}^{+\infty}x^r\sum_{k=0}^{r}(-1)^k\binom{r+n}{k+n}\\&=\sum_{k=0}^{+\infty}(-1)^k\sum_{r=k}^{+\infty}\binom{r+n}{k+n}x^r=\sum_{k=0}^{+\infty}(-1)^kx^k\sum_{r=0}^{+\infty}\binom{r+k+n}{k+n}x^r\\&=\sum_{k=0}^{+\infty}(-1)^kx^k(1-x)^{-n-k-1}=(1-x)^{-n-1}\sum_{k=0}^{+\infty}(-1)^kx^k(1-x)^{-k}\\&=(1-x)^{-n},\end{aligned} \tag{41}$$

it holds

$$\binom{n+r-1}{r}=\sum_{k=0}^{r}(-1)^k\binom{r+n}{k+n}. \tag{42}$$

According to (39) and (42), we need to show

$$\sum_{k=r}^{m-n}\binom{n+k-1}{k}\binom{k}{r}=\binom{m}{m-n-r}\binom{n+r-1}{r}. \tag{43}$$

Noticing that

$$\begin{aligned}&\sum_{k=r}^{m-n}\binom{n+k-1}{k}\binom{k}{r}=\sum_{k=0}^{m-n-r}\binom{n+k+r-1}{k+r}\binom{k+r}{r}\\&=\sum_{k=0}^{m-n-r}\frac{(k+r+n-1)!}{(n-1)!k!r!}=\binom{n+r-1}{r}\sum_{k=0}^{m-n-r}\binom{n+k+r-1}{k},\end{aligned} \tag{44}$$

this reduces to show that

$$\binom{m}{m-n-r}=\sum_{k=0}^{m-n-r}\binom{n+k+r-1}{k}. \tag{45}$$

Writing $i=n+r$, the above expression is

$$\binom{m}{m-i}=\sum_{k=0}^{m-i}\binom{k+i-1}{k}=\sum_{k=0}^{m-i}\binom{k+i-1}{i-1}=\sum_{j=i-1}^{m-1}\binom{j}{i-1}. \tag{46}$$

In order to prove this, we use the formula

$$\sum_{j=n}^{N}\binom{j}{n}=\binom{N+1}{n+1} \tag{47}$$

in a handbook of mathematics [33]. Setting $N=m-1,\ n=i-1$ in above formula gives

$$\sum_{j=r-1}^{m-1}\binom{j}{i-1}=\binom{m}{i}=\binom{m}{m-i}.\quad\square \tag{48}$$

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**Remark 1.** According to Theorem 2, the Euler transform $\mathscr{E}(q)$ is only a special case of the so-called homotopy transform $\mathscr{T}(\hbar, \vec{\alpha}, \vec{\beta})$ defined by (30) when $\hbar = -q$, $\alpha_{1,1} = \beta_{1,1} = 1$ and $\alpha_{1,k} = \beta_{1,k} = 0$ $(k > 1)$, corresponding to the two simplest deformation functions $A(p) = B(p) = p$.

Here, we would like to emphasize two points. First, the Euler transform is only a special case of the so-called homotopy transformation. Thus, the homotopy transform is more general and thus should be more powerful. Secondly, Euler transform is widely used to accelerate convergence of a series or to make a divergent series convergent. Thus, the homotopy transform provides us with a new but more general way to accelerate convergence of a series or to make a divergent series convergent.

## 4. Relation between the homotopy analysis method and Euler transform

In this section, we use one simple example to show that the so-called homotopy transform defined by (30) can be obtained in the frame of the homotopy analysis method.

For the simplicity, let us consider a nonlinear ordinary differential equation

$$u'(x) + u(x)\left[1 - \frac{1}{2}u(x)\right] = 0 \tag{49}$$

subject to the boundary condition

$$u(0) = 1. \tag{50}$$

This equation has the closed-form solution $u(x) = 2/(1 + e^x)$.

Let $\hbar \neq 0$ denote a non-zero auxiliary parameter, $p \in [0, 1]$ the embedding parameter, and $\bar{A}(p), \bar{B}(p)$ the deformation functions satisfying

$$\bar{A}(0) = \bar{B}(0) = 0,\ \bar{A}(1) = \bar{B}(1) = 1, \tag{51}$$

and their Maclaurin series $A(p) = \sum_{k=1}^{+\infty} \bar{\alpha}_{1,k}\, p^k$ and $B(p) = \sum_{k=1}^{+\infty} \bar{\beta}_{1,k}\, p^k$ are convergent at $p = 1$, respectively. Define the nonlinear operator

$$\mathscr{N} u = \frac{du}{dx} + u\left(1 - \frac{1}{2}u\right). \tag{52}$$

We construct a family of equations

$$[1 - \bar{B}(p)]\mathscr{L}[\theta(x,p) - u_0(x)] = \hbar \bar{A}(p)\mathscr{N}[\theta(x,p)], \tag{53}$$

subject to the boundary condition

$$\theta(0, p) = 1, \tag{54}$$

where $\mathscr{L}$ is a properly chosen *auxiliary linear operator* satisfying

$$\mathscr{L}(0) = 0, \tag{55}$$

and $u_0(x)$ is an initial guess satisfying the boundary condition $u_0(0) = 1$. When $p = 0$, according to (51), we have from Eqs. (53) and (54) that

$$\theta(x, 0) = u_0(x). \tag{56}$$

When $p = 1$, Eqs. (53) and (54) are exactly the same as the original Eqs. (49) and (50), respectively, thus it holds

$$\theta(x, 1) = u(x). \tag{57}$$

Assume that $\hbar, \bar{A}(p), \bar{B}(p)$ are so properly chosen that the solution $\theta(x, p)$ of Eqs. (53) and (54) exists for all $p \in [0, 1]$. Thus, as the embedding parameter $p$ increases from 0 to 1, the solution $\theta(x, p)$ of Eqs. (53) and (54) varies continuously from the initial guess $u_0(x)$ to the solution $u(x)$ of the original Eqs. (49) and (50). This continuous variation is called deformation in topology. So, Eqs. (53) and (54) are called *the zeroth-order deformation equations*.

Note that we have great freedom to choose $\mathscr{L}, u_0(x), \hbar, \bar{A}(p)$ and $\bar{B}(p)$. Assume that all of them are so properly chosen that the Maclaurin series of $\theta(x, p)$ about $p$, i.e.

$$\theta(x, p) \sim u_0(x) + \sum_{k=1}^{+\infty} u_k(x)\, p^k, \tag{58}$$

exists and besides is convergent at $p = 1$, where

$$u_k(x) = \frac{1}{k!} \frac{\partial^k \theta(x, p)}{\partial p^k}\bigg|_{p=0}.$$

Thus, we have by (57) the series solution

$$u(x) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x). \tag{59}$$

The above expression gives a relationship between the initial guess $u_0(x)$ and the solution $u(x)$ of the original Eqs. (49) and (50) via the unknown terms $u_m(x)$.

Differentiating the zeroth-order deformation Eqs. (53) and (54) $m$ times ($m \geqslant 1$) with respect to $p$, dividing them by $m!$ and then setting $p = 0$, we have the so-called *mth-order deformation equations*

$$\mathscr{L}\left[u_m(x) - \sum_{k=1}^{m-1} \bar{\beta}_{1,k}\, u_{m-k}(x)\right] = \hbar \sum_{k=1}^{m-1} \bar{\alpha}_{1,m-k}\, R_k(x), \tag{60}$$

subject to the boundary condition

$$u_m(0) = 0, \tag{61}$$

where

$$R_k(x) = \frac{1}{k!} \frac{\partial^k \mathscr{N}[\theta(x,p)]}{\partial p^k}\bigg|_{p=0} = u_k'(x) + u_k(x) - \frac{1}{2}\sum_{i=0}^{k} u_i(x) u_{k-i}(x). \tag{62}$$

The partial sum

$$u_0(x) + \sum_{m=1}^{M} u_m(x) \tag{63}$$

gives the $M$th-order approximation of the considered nonlinear problem.

Different from perturbation techniques, the above approach is independent of any small/large physical parameters and thus is more general. Besides, it provides us great freedom to choose the initial guess and the auxiliary linear operator $\mathscr{L}$ so that we can use different types of base functions to approximate the solution. More importantly, the freedom on the choice of the auxiliary parameter $\hbar$ and the two deformation functions $\bar{A}(p)$ and $\bar{B}(p)$ in the zeroth-order deformation Eqs. (53) and (54) provides us a convenient way to ensure the convergence of the solution series. Here, using this kind of freedom, we show that the homotopy transform (6) can be obtained in the frame of the homotopy analysis method in some special cases.

Let $A(p)$ and $B(p)$ be two deformation functions satisfying $A(0) = B(0) = 0$ and $A(1) = B(1) = 1$, and their Maclaurin series $A(p) = \sum_{k=1}^{+\infty} \alpha_{1,k}\, p^k$ and $B(p) = \sum_{k=1}^{+\infty} \beta_{1,k}\, p^k$ are convergent at $p = 1$. Then, according to the definition (51), it is obvious that

$$\bar{A}(p) = A(p) \tag{64}$$

and

$$\bar{B}(p) = B(p) + \hbar[B(p) - A(p)] \tag{65}$$

are also deformation functions, whose Maclaurin series are

$$\bar{A}(p) = \sum_{k=1}^{+\infty} \alpha_{1,k} p^k \tag{66}$$

and

$$\bar{B}(p) = \sum_{k=1}^{+\infty} [(1+\hbar)\beta_{1,k} - \hbar \alpha_{1,k}] p^k, \tag{67}$$

respectively. Moreover, we choose

$$u_0(x) = 1 \tag{68}$$

as our initial guess and

$$\mathscr{L} = \frac{d}{dx} \tag{69}$$

as our auxiliary linear operator. Then, the zeroth-order deformation Eq. (53) becomes

$$[1 - (1+\hbar)B(p) + \hbar A(p)] \frac{\partial \theta(x,p)}{\partial x} = \hbar A(p)\left\{\frac{\partial \theta(x,p)}{\partial x} + \theta(x,p)\left[1 - \frac{1}{2}\theta(x,p)\right]\right\} \tag{70}$$

and the corresponding $m$th-order ($m \geqslant 1$) deformation equation is

$$\frac{d}{dx}\left\{u_m(x)-\sum_{k=1}^{m-1}[(1+\hbar)\beta_{1,m-k}-\hbar\alpha_{1,m-k}]u_k(x)\right\}=\hbar\sum_{k=1}^{m-1}\alpha_{1,m-k}\left[u_k'(x)+u_k(x)-\frac{1}{2}\sum_{i=0}^{k}u_i(x)u_{k-i}(x)\right], \tag{71}$$

with the corresponding boundary condition

$$u_m(0)=0. \tag{72}$$

The solution of the high-order deformation Eqs. (71) and (72) is given by the following recurrence formula

$$u_m(x)=\sum_{k=1}^{m-1}[(1+\hbar)\,\beta_{1,m-k}-\hbar\,\alpha_{1,m-k}]u_k(x)+\hbar\sum_{k=1}^{m-1}\alpha_{1,m-k}\int_0^x\left[u_k'(x)+u_k(x)-\frac{1}{2}\sum_{i=0}^{k}u_i(x)u_{k-i}(x)\right]dx \tag{73}$$

Thus, using the initial guess (68) and above recurrence formula, we can get the high-order approximations by means of symbolic computation software. In surprise, it is found that the corresponding $m$th-order approximation reads

$$u_0(x)+\sum_{k=1}^{m}u_k(x)=1+\sum_{k=1}^{m}(a_k\,x^k)T_{m,k}(\hbar,\vec{\alpha},\vec{\beta}), \tag{74}$$

where $T_{m,k}(\hbar,\vec{\alpha},\vec{\beta})$ is exactly given by (6) under the same definitions (3)–(5), and the coefficient $a_k$ is given by the Taylor series $\sum_{k=0}^{+\infty}a_k\,x^k$ of the exact solution $u(x)=2/(1+e^x)$ of Eqs. (49) and (50).

We can prove the correctness of (74) in another way. Notice that the zeroth-order deformation Eq. (70) can be rewritten as

$$\frac{\partial\theta(x,p)}{\partial\left[\left(\frac{-\hbar A(p)}{1-(1+\hbar)B(p)}\right)x\right]}+\theta(x,p)\left[1-\frac{1}{2}\theta(x,p)\right]=0, \tag{75}$$

whose solution, satisfying the boundary condition (54), is exactly

$$\theta(x,p)=\frac{2}{1+\exp\left[\frac{-\hbar A(p)x}{1-(1+\hbar)B(p)}\right]}. \tag{76}$$

Similarly as showed in Section 2, expanding $\theta(x,p)$ defined above in power series of $p$ and then setting $p=1$, we get

$$u(x)=\theta(x,1)=1+\lim_{M\to+\infty}\sum_{k=1}^{M}(a_kx^k)T_{m,k}(\hbar,\vec{\alpha},\vec{\beta}), \tag{77}$$

where $T_{m,k}(\hbar,\vec{\alpha},\vec{\beta})$ is exactly the same as (6), and $a_k$ is given by the Taylor series $\sum_{k=0}^{+\infty}a_k\,x^k$ of the exact solution $u(x)=2/(1+e^x)$.

Therefore, the so-called homotopy transform described in Section 3 can be indeed derived in the frame of the homotopy analysis method in some special cases. As proved in Section 3, the famous Euler transform $\mathscr{E}(q)$ is only a special case of the homotopy transform $\mathscr{T}(\hbar,\vec{\alpha},\vec{\beta})$ in case of $\hbar=-q$ and $\alpha_{1,1}=\beta_{1,1}=1$ and $\alpha_{1,k}=\beta_{1,k}=0$ for $k>1$, corresponding to the simplest deformation functions $A(p)=B(p)=p$. Thus, for some special choices of the initial guess and the auxiliary linear operator, the homotopy analysis method in case of $A(p)=B(p)=p$ might be sometimes equivalent to the Euler transform.

On one side, this fact explains why the convergence of solution series given by the homotopy analysis method can be guaranteed, because the Euler transform is widely applied to accelerate the convergence of a series or to make a divergent series convergent. On the other side, it should be emphasized that the homotopy analysis method is much more general than the Euler transform, because one has great freedom to choose not only different types of deformation functions $A(p)$ and $B(p)$, but also the auxiliary linear operator $\mathscr{L}$ and the initial guess. Note that the homotopy transform (30) is obtained in the frame of the homotopy analysis method by using the special initial guess (68) and the special auxiliary linear operator (69) for the considered example. However, by means of the homotopy analysis method, we have great freedom to choose other types of initial guess and auxiliary linear operators. For example, if the auxiliary linear operator $\mathscr{L}u=u'+\lambda u$ and the initial guess $u_0(x)=\exp(-\lambda\,x)$ are chosen for the considered simple example, where $\lambda>0$ is the second auxiliary parameter, we can obtain approximations expressed by exponential base functions

$$\{\exp(-\lambda x),\exp(-2\lambda x),\exp(-3\lambda x),\ldots\}$$

Obviously, such kind of approximations contain two non-zero auxiliary parameters $\hbar$ and $\lambda$, and thus is certainly different from the Euler transform that has only one auxiliary parameter $q$. In general, if approximations given by the homotopy analysis method have more than one auxiliary parameters, it is certainly not equivalent to the Euler transform. For example, for a nonlinear problem governed by $n$ coupled equations, the solution series given by the homotopy analysis method may contain $n$ different auxiliary parameters $\hbar_1,\hbar_2,\ldots,\hbar_n$ even in case of $A(p)=B(p)=p$, which is certainly *not* equivalent to the Euler transform.

In summary, the famous Euler transform is equivalent to the homotopy analysis method for some special choices of the initial guess and the auxiliary linear operators in case of the simplest deformation functions $A(p)=B(p)=p$ and when there is only one auxiliary parameter $\hbar$. But, dependent upon (at least) one non-zero auxiliary parameter $\hbar$ and two convergent

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series $\sum_{k=0}^{+\infty}\alpha_{1,k}$ and $\sum_{k=0}^{+\infty}\beta_{1,k}$, the homotopy analysis method is more general and thus more powerful than the Euler transform.

However, it is still an open question how to choose better auxiliary parameter $\hbar$ and better deformation functions $A(p)$ and $B(p)$, corresponding to the two convergent series $\sum_{k=0}^{+\infty}\alpha_{1,k}$ and $\sum_{k=0}^{+\infty}\beta_{1,k}$, so that the corresponding series solution given by the homotopy analysis method converges faster. So, some pure mathematical studies are needed in future.

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