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Solving the nonlinear periodic wave problems with the Homotopy Analysis Method

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Abstract

An analytic technique, namely the Homotopy Analysis Method (HAM), is applied to solve the nonlinear mKdV equation. Solutions for periodic waves are given and compared with the exact ones, which shows the validity of the HAM for the nonlinear periodic wave problems.

Keywords: mKdV equation; Homotopy Analysis Method; Nonlinear wave problem

1. Introduction

Homotopy Analysis Method (HAM [1]) is an analytic technique for nonlinear problems. This method has been successfully applied to many nonlinear problems in engineering and science, such as the magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet [2], nonlinear progressive waves in deep water [3], free oscillations of positively damped systems with algebraically decaying amplitude [4], free oscillations of self-excited systems [5], similarity boundary layer equations [6]. All of these successful applications verified the validity, effectiveness and flexibility of the HAM. For more details, we refer the reader to Liao [1,7].

In this paper, we apply the HAM to the mKdV equation

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0, \tag{1}$$

which describes the motions of waves in nonlinear optics, plasma or fluids. Periodic solutions for this equation are given and verified by the exact ones in terms of Jacobi elliptic function [8,9]. The validity and effectiveness of the HAM in solving the nonlinear periodic wave problems are shown.

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2. Mathematical formulation

Consider the travelling wave solutions of Eq. (1). Under the transformation

$$u(x,t) = Af(\theta), \quad \theta = kx - \omega t, \tag{2}$$

where A is the wave amplitude, k the wave number, and ω the angular frequency, the mKdV Eq. (1) becomes

$$-cf' + \gamma f^2 f' + \mu f''' = 0, \tag{3}$$

where primes denote derivatives with respect to θ, and

$$c = \frac{\omega}{k}, \qquad \gamma = \alpha A^2, \qquad \mu = \beta k^2. \tag{4}$$

In this section, we will give analytic solutions to Eq. (3) by means of HAM.

Due to the periodicity of the problem, the solution $f(\theta)$ of Eq. (3) can be expressed by the Fourier Sine Series, if $f(\theta)$ is an odd function, or by the Fourier Cosine Series, if $f(\theta)$ is an even function. This provides us with the Rule of Solution Expression (RSE), which is the cornerstone of the HAM, and will be discussed respectively.

2.1. Solution expressed by Sine Series

In this case, the solution can be expressed by

$$f(\theta) = \sum_{m=1}^{+\infty} a_m \sin(m\theta), \tag{5}$$

where a_m $(m = 1, 2, \ldots, +\infty)$ are coefficients. This provides us with the Rule of Solution Expression. Choosing $f_0(\theta) = \sin\theta$ as the initial guess of $f(\theta)$, and

$$L[\phi(\theta,q)] = \frac{\partial^3\phi(\theta,q)}{\partial\theta^3} + \frac{\partial\phi(\theta,q)}{\partial\theta} \tag{6}$$

as the auxiliary linear operator, we construct the zeroth order deformation equation

$$(1-q)L[F(\theta,q) - f_0(\theta)] = qhN[F(\theta,q), C(q)], \tag{7}$$

where q is an embedding parameter, h a non-zero auxiliary parameter, and

$$N[F(\theta,q), C(q)] = -C(q)F'(\theta,q) + \gamma F^2(\theta,q)F'(\theta,q) + \mu F'''(\theta,q), \tag{8}$$

where primes denote derivatives with respect to θ. It is seen from Eq. (7) that as the parameter q increases from 0 to 1, the solution $F(\theta,q)$ varies from $f_0(\theta)$ to $f(\theta)$, so does the $C(q)$ from c_0, the initial guess of c, to c. If this continuous variation is smooth enough, the Maclaurin's series with respect to q can be constructed for $F(\theta,q)$ and $C(q)$ respectively, and further, if these two series are convergent at $q = 1$, we then have

$$f(\theta) = f_0(\theta) + \sum_{m=1}^{+\infty} f_m(\theta), \tag{9}$$

$$c = c_0 + \sum_{m=1}^{+\infty} c_m, \tag{10}$$

where

$$f_m(\theta) = \frac{1}{m!} \frac{\partial^m F(\theta, q)}{\partial q^m}\bigg|_{q=0}, \tag{11}$$

$$c_m = \frac{1}{m!} \frac{\partial^m C(q)}{\partial q^m}\bigg|_{q=0}, \tag{12}$$

are called the mth order deformation derivatives.

Differentiating Eq. (7) m times with respect to q then setting $q = 0$ and finally dividing them by $m!$, we gain the mth order deformation equations for $f_m(\theta)$

$$L[f_m(\theta) - \chi_m f_{m-1}(\theta)] = \hbar R_m(\theta), \quad m \geq 1, \tag{13}$$

where

$$\begin{aligned} R_m(\theta) &= \frac{1}{(m-1)!} \frac{\partial^{m-1} N[F(\theta, q), C(q)]}{\partial q^{m-1}}\bigg|_{q=0} \\ &= \gamma \sum_{n=0}^{m-1} \left(\sum_{k=0}^{n} f_k(\theta) f_{n-k}(\theta) \right) f'_{m-1-n}(\theta) - \sum_{n=0}^{m-1} c_n f'_{m-1-n}(\theta) + \beta f'''_{m-1}(\theta), \end{aligned} \tag{14}$$

and

$$\chi_m = \begin{cases} 1, & m > 1, \\ 0, & m = 1. \end{cases} \tag{15}$$

It should be emphasized that $R_m(\theta)$ is function of $f_k(\theta)$ and c_k, where $k = 0, 1, 2, \ldots, m - 1$. They are known when solving $f_m(\theta)$, except for c_{m-1}. Furthermore, $R_m(\theta)$ can be expressed by

$$R_m(\theta) = \sum_{n=1}^{m+1} \xi_{m,n} \cos[(2n - 1)\theta], \tag{16}$$

where $\xi_{m,n}$ is a coefficient. Due to the Rule of Solution Expression (5), the solution of Eq. (13) should *not* contain the so-called secular term $\theta \cos\theta$. To ensure this, $R_m(\theta)$ should not contain the term $\cos\theta$, i.e. the coefficient of $\cos\theta$ must be zero. This leads to an algebraic equation

$$\xi_{m,1} = 0, \tag{17}$$

which can be used to determine c_{m-1}.

The general solution of Eq. (13) is

$$f_m(\theta) = -h \sum_{n=2}^{m+1} \frac{\xi_{m,n}}{4n(n-1)(2n-1)} \sin[(2n-1)\theta] + \chi_m f_{m-1}(\theta) + C_1 + C_2 \cos\theta + C_3 \sin\theta, \tag{18}$$

where C_1, C_2, and C_3 are the integral constants. According to the Rule of Solution Expression (5), we have

$$C_1 = C_2 = 0. \tag{19}$$

To ensure the amplitude of $f(\theta)$ to be 1, we demand

$$f_m(\tfrac{1}{2}\pi) - f_m(\tfrac{3}{2}\pi) = 0, \tag{20}$$

which determines the constant C_3. We solve Eqs. (13) and (17) for $m = 1, 2, 3, \ldots$, successively, and at the Mth order approximation we have the analytic solution

$$f(\theta) \approx \sum_{m=0}^{M} f_m(\theta), \tag{21}$$

$$c \approx \sum_{m=0}^{M} c_m. \tag{22}$$

2.2. Solution expressed by Cosine Series

In this case, the solution $f(\theta)$ of Eq. (3) can be expressed by the Cosine series $\cos(m\theta)$, where $m = 0, 1, 2, \ldots, +\infty$. Note that, $f(\theta)$ contains a constant term, which is the main difference between this case and the previous one. Integrating Eq. (3) once respect to θ, and letting the integral constant to be zero, we have

$$\mu f'' - cf + \tfrac{1}{3}\gamma f^3 = 0. \tag{23}$$

Under the transformation

$$f(\theta) = \delta + \lambda g(\theta), \tag{24}$$

where δ is a constant, and λ the amplitude of the periodic function $f(\theta)$, Eq. (23) becomes

$$\mu\lambda g''(\theta) - c[\delta + \lambda g(\theta)] + \tfrac{1}{3}\gamma[\delta + \lambda g(\theta)]^3 = 0, \tag{25}$$

where the constant term δ, the amplitude λ, the wave velocity c, and the periodic function $g(\theta)$ are unknowns to be determined under the frame of HAM. The solution of (25) can be expressed by

$$g(\theta) = \sum_{m=1}^{+\infty} b_m \cos(m\theta), \tag{26}$$

where b_m ($m = 1, 2, \ldots, +\infty$) are coefficients. This provides us with the Rule of Solution Expression. Choosing $g_0(\theta) = \cos\theta$ as the initial guess of $g(\theta)$, and

$$L[\phi(\theta, q)] = \frac{\partial^2\phi(\theta, q)}{\partial\theta^2} + \phi(\theta, q) \tag{27}$$

as the linear operator, we construct the zeroth order deformation equation

$$(1 - q)L[G(\theta, q) - g_0(\theta)] = \hbar q N[G(\theta, q), C(q), \Delta(q), \Lambda(q)], \tag{28}$$

subject to the restrictions

$$G(0, q) - G(\pi, q) = 2, \tag{29}$$

$$\Delta(q) + \Lambda(q)G(0, q) = 1, \tag{30}$$

with definition

$$N[G(\theta, q), C(q), \Delta(q), \Lambda(q)] = \mu\Lambda G'' - C(\Delta + \Lambda G) + \tfrac{1}{3}\gamma(\Delta + \Lambda G)^3, \tag{31}$$

where primes denote derivatives with respect to θ. The corresponding mth order deformation equation is

$$L[g_m(\theta) - \chi_m g_{m-1}(\theta)] = hR_m(\theta), \tag{32}$$

subject to the restrictions

$$g_m(0) - g_m(\pi) = 0, \tag{33}$$

$$\delta_m + \sum_{n=0}^{m} \lambda_n g_{m-n}(0) = 0, \tag{34}$$

for $m \geq 1$, where $R_m(\theta)$ is a function of $g_k(\theta)$, c_k, δ_k and λ_k $(k = 0, 1, 2, \ldots, m-1)$ which are the kth order deformation derivatives corresponding to $G(\theta, q)$, $C(q)$, $\Delta(q)$ and $\Lambda(q)$, respectively. Note that $R_m(\theta)$ contains three unknowns, i.e. c_{m-1}, δ_{m-1}, and λ_{m-1}, when solving $g_m(\theta)$.

It is found that $R_m(\theta)$ can be expressed by

$$R_m(\theta) = \sum_{n=0}^{2m+1} \xi_{m,n} \cos(n\theta), \tag{35}$$

where $\xi_{m,n}$ is a coefficient. Under the Rule of Solution Expression (26), the solution $g_m(\theta)$ of Eq. (32) should not contain the constant term and the so-called secular term $\theta \sin\theta$. To ensure this, $R_m(\theta)$ should not contain the constant term and the term $\cos\theta$, which leads to two algebraic equations

$$\xi_{m,0} = 0, \qquad \xi_{m,1} = 0, \tag{36}$$

to determine c_{m-1}, δ_{m-1} and λ_{m-1} with the aid of Eq. (34). When $m = 1$, this set of algebraic equations is

$$\tfrac{1}{3}\gamma\delta_0^2 + \tfrac{1}{2}\gamma\lambda_0^2 - c_0 = 0, \tag{37}$$

$$\tfrac{1}{4}\gamma\lambda_0^2 + \gamma\delta_0^2 - c_0 - \mu = 0, \tag{38}$$

$$\lambda_0 + \delta_0 = 1, \tag{39}$$

which is nonlinear, and from which we get

$$c_0 = \tfrac{1}{5}(11\gamma + 10\mu - 4\sqrt{6\gamma^2 + 15\gamma\mu}), \tag{40}$$

$$\delta_0 = \tfrac{1}{5}(-3 + 2\sqrt{6 + 15\mu/\gamma}), \tag{41}$$

$$\lambda_0 = \tfrac{2}{5}(4 - \sqrt{6 + 15\mu/\gamma}), \tag{42}$$

as the initial guess of c, δ and λ. However, this set of algebraic equations is always linear when $m \geq 2$, and can be solved easily.

The general solution of Eq. (32) is

$$g_m(\theta) = -\hbar \sum_{n=2}^{2m+1} \frac{\xi_{m,n}\cos(n\theta)}{n^2 - 1} + \chi_m f_{m-1}(\theta) + C_1 \sin\theta + C_2 \cos\theta, \tag{43}$$

where C_1, C_2 are the integral constants. According to the Rule of Solution expression (26), we have $C_1 = 0$. The constant C_2 is determined by (33). In this way, we get $g_m(\theta)$, c_{m-1}, δ_{m-1} and λ_{m-1} for $m = 1, 2, 3, \ldots$, successively, and at the *M*th order approximation, we have the analytic solutions of Eq. (23):

$$f(\theta) = \delta + \lambda g(\theta) \approx \sum_{m=0}^{M} \delta_m + \left(\sum_{m=0}^{M} \lambda_m\right)\left(\sum_{m=0}^{M} g_m(\theta)\right), \tag{44}$$

$$c \approx \sum_{m=0}^{M} c_m. \tag{45}$$

3. Validation of the solutions

The procedure described above can be realized easily with the aid of symbol calculation software, such as MATHEMATICA. In this section, we verify our analytic solutions with the exact ones in terms of Jacobi elliptic functions [8,9].

In the frame of HAM, the periodic solution is expressed as a series of sine or cosine functions. Note that our solution series contains the parameter $\hbar$, which provides us with a simply way to adjust and control the convergence of the solution series. In general, by means of the so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series. Fig. 1 shows the wave velocity under different $\hbar$, compared with the exact value 1.6682, in case of $\gamma = 5$ and $\mu = -0.1$. It is seen that convergent results can be obtained when $1 < \hbar < 3$. Thus, we can choose an appropriate value for $\hbar$ in this range to get convergent solution of the wave velocity. When $\hbar = 2$, our solution of c is 1.66866, which is very close to the exact value. In this case, our analytic solution of $f(\theta)$ expressed by sine series agrees well with the exact one in terms of Jacobi elliptic *sine* function, as shown in Fig. 2. Note that, when $\gamma = 5$ and $\mu = -0.1$, the modulus of the solution expressed by Jacobi elliptic sine functions is 0.999083, which means the wave tends to a solitary wave [8,9].

Fig. 3 shows the 20th order of approximation of $f(\theta)$ expressed by sine series, comparing with the exact one in terms of Jacobi elliptic *cosine* function, in case of $\gamma = 5$, $\mu = 0.1$ and $h = -2$. In this case, the wave velocity given by HAM is 0.831337, also agrees well with the exact value 0.831803. Fig. 4 compares the analytic solution

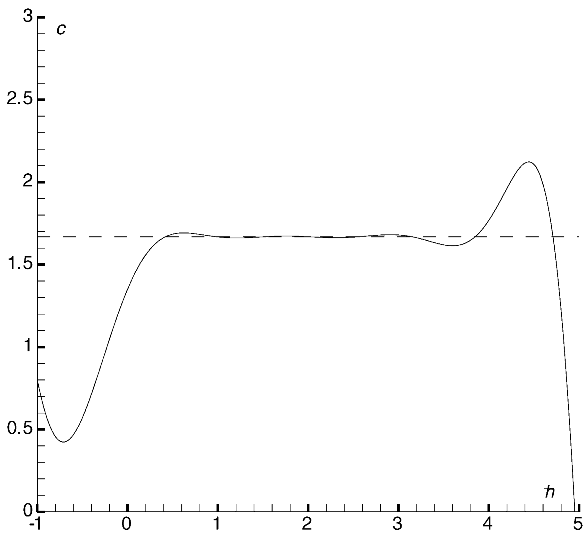

Fig. 1. Wave velocity c under 20th order of approximation with different $\hbar$, compared with the exact value 1.6682, in case of $\gamma = 5$, $\mu = -0.1$. Solid line: solution given by HAM; dotted line: exact value.

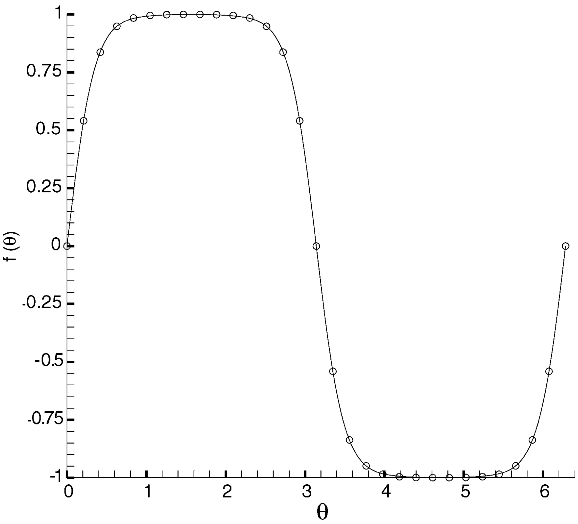

Fig. 2. Solution of $f(\theta)$ under 20th order of approximation, compared with the exact one in case of $\gamma = 5$, $\mu = -0.1$, $\hbar = 2$. Solid line: solution given by HAM; circle: exact solution in terms of Jacobi elliptic sine function.

expressed by cosine series at 15th order of approximation with the exact one in terms of the *third kind* of Jacobi elliptic function, in case of $\gamma = 1$, $\mu = 0.1$, $\hbar = -11$. In this case, the wave velocity *c* given by HAM is 0.167473, which is the same as the exact value.

In this paper, we compared our analytic solutions with the exact ones. However, there are cases where we can not find the later. If so, we can substitute the solutions given by HAM into the equations considered and evaluate the errors to check the convergence of the solutions. Fig. 5 shows the error of Eq. (23), in case of $\gamma = 1$, $\mu = 0.1$, $\hbar = -11$. It is seen that $f(\theta)$ can be well approximated by the analytic solutions given by HAM.

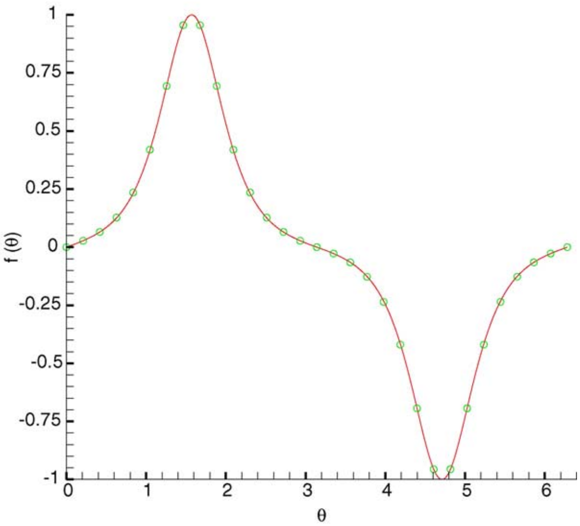

Fig. 3. Solution of $f(\theta)$ under 20th order of approximation, compared with the exact one in case of $\gamma = 5$, $\mu = 0.1$, $\hbar = -2$. Solid line: solution given by HAM; circle: exact solution in terms of Jacobi elliptic cosine function.

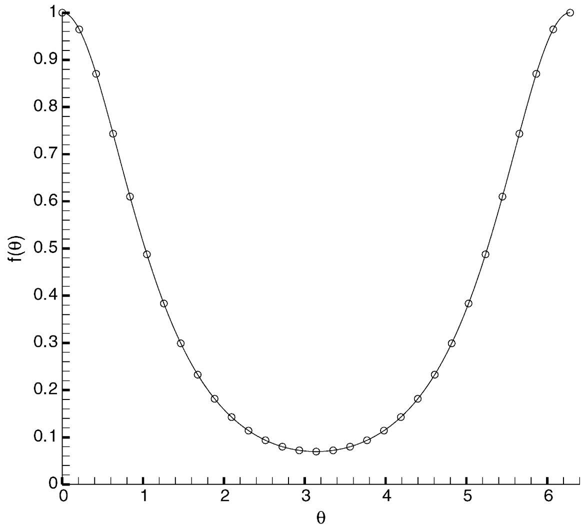

Fig. 4. Solution of $f(\theta)$ under 15th order of approximation, compared with the exact one in case of $\gamma = 1$, $\mu = 0.1$, $\hbar = -11$. Solid line: solution given by HAM; circle: exact solution in terms of the third kind of Jacobi elliptic function.

It should be pointed out that, although few cases of γ and μ are illustrated in this paper, our analytic solutions are uniformly valid for most of γ and μ, which is ensured by the parameter $\hbar$. In addition, this paper shows the comparisons of our analytic solutions with the exact ones in terms of Jacobi elliptic sine function, Jacobi elliptic cosine function, and the third kind of Jacobi elliptic function. However, the mKdV Eq. (1) has also solutions in terms of Jacobi elliptic cs function [8,9], which has singularities and can not be approximated by the method described in this paper.

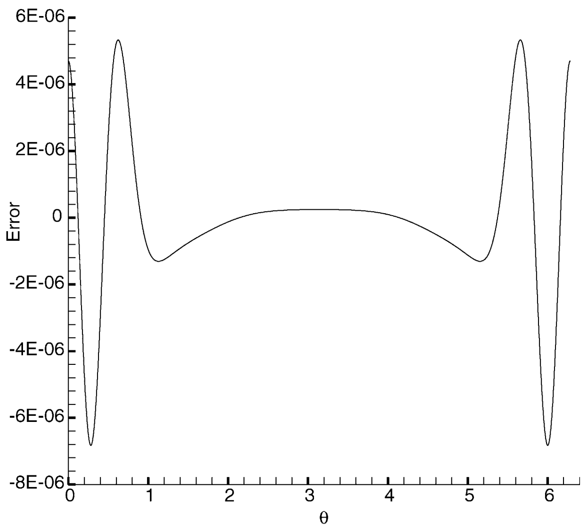

Fig. 5. Error evaluated by substituting our analytic solutions of $f(\theta)$ under 15th order of approximation into Eq. (23), in case of $\gamma = 1$, $\mu = 0.1$, $\hbar = -11$.

4. Conclusions

In this paper, we applied the Homotopy Analysis Method (HAM [1]) to give the analytic periodic wave solutions to the mKdV equation (1), and verified the validity of the solutions by comparisons with the exact ones in terms of Jacobi elliptic functions ([8,9]). This indicates that the Homotopy Analysis Method is valid for nonlinear wave problems with periodic solutions.

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