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A one-step optimal homotopy analysis method for nonlinear differential equations

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ABSTRACT

In this paper, a one-step optimal approach is proposed to improve the computational efficiency of the homotopy analysis method (HAM) for nonlinear problems. A generalized homotopy equation is first expressed by means of a unknown embedding function in Taylor series, whose coefficient is then determined one by minimizing the square residual error of the governing equation. Since at each order of approximation, only one algebraic equation with one unknown variable is solved, the computational efficiency is significantly improved, especially for high-order approximations. Some examples are used to illustrate the validity of this one-step optimal approach, which indicate that convergent series solution can be obtained by the optimal homotopy analysis method with much less CPU time. Using this one-step optimal approach, the homotopy analysis method might be applied to solve rather complicated differential equations with strong nonlinearity.

1. Introduction

The homotopy analysis method (HAM) [1–6] was proposed to get analytic approximations of highly nonlinear equations. The homotopy is a basic concept in topology, and has been widely applied in pure mathematics and numerical algorithms. In 1992, Liao [1] first used the concept of homotopy to obtain analytic approximations of nonlinear equations $\mathscr{N}[u(\mathbf{r})] = 0$ by means of constructing a one-parameter family of equations (called the zeroth-order deformation equation)

$$(1-q)\mathscr{L}[\phi(\mathbf{r};q) - u_0(\mathbf{r})] = q\mathscr{N}[\phi(\mathbf{r};q)], \tag{1}$$

where $q \in [0, 1]$ is an embedding parameter, $\mathscr{N}$ is a nonlinear operator, $u(\mathbf{r})$ is a unknown function, $u_0(\mathbf{r})$ is a guess approximation, $\mathbf{r}$ denotes independent variable(s), respectively. Obviously, we have $\phi(\mathbf{r};0) = u_0(\mathbf{r})$ when $q = 0$, and $\phi(\mathbf{r};1) = u(\mathbf{r})$ when $q = 1$, respectively. The Taylor series of $\phi(\mathbf{r};q)$ with respect to the embedding parameter q reads

$$\phi(\mathbf{r};q) = u_0(\mathbf{r}) + \sum_{m=1}^{+\infty} u_m(\mathbf{r})q^m, \tag{2}$$

where

$$u_m(\mathbf{r}) = \frac{1}{m!}\frac{\partial^m \phi(\mathbf{r};q)}{\partial q^m}\bigg|_{q=0}. \tag{3}$$

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Assuming that the Taylor series (2) is convergent at $q = 1$, we have the series solution

$$u(\mathbf{r}) = u_0(\mathbf{r}) + \sum_{m=1}^{+\infty} u_m(\mathbf{r}). \tag{4}$$

However, the above approach breaks down if the Taylor series (2) diverges at $q = 1$.

To overcome this disadvantage, Liao [3] introduced in 1997 a nonzero auxiliary parameter $\hbar$, which is now called the convergence-control parameter [6], to construct such a two-parameter family of equations (i.e. the zeroth-order deformation equation)

$$(1-q)\mathscr{L}[\phi(\mathbf{r};q) - u_0(\mathbf{r})] = \hbar q \mathscr{N}[\phi(\mathbf{r};q)]. \tag{5}$$

Note that the solution $\phi(\mathbf{r};q)$ of the above equation is not only dependent upon the embedding parameter q but also the convergence-control parameter $\hbar$. So, the term u_m given by (3) is also dependent upon $\hbar$ and therefore the convergence region of the Taylor series (2) is influenced by $\hbar$. Thus, the auxiliary parameter $\hbar$ provides us a convenient way to ensure the convergence of the Taylor series (2) at $q = 1$, as illustrated in [3]. The introduction of the convergence-control parameter $\hbar$ greatly improves the homotopy analysis method in theory, as shown by Liao and Tan [5], Abbasbandy [7], Sajid and Hayat [8], and Liang and Jeffrey [9].

In 1999, Liao [4] further generalized the homotopy analysis method by constructing such a zeroth-order deformation equation

$$[(1-B(q)]\mathscr{L}[\phi(\mathbf{r};q) - u_0(\mathbf{r})] = \hbar A(q) \mathscr{N}[\phi(\mathbf{r};q)], \tag{6}$$

where $A(q)$ and $B(q)$ are two analytic functions satisfying

$$A(0) = B(0) = 0, \quad A(1) = B(1) = 1. \tag{7}$$

Obviously, q is only a special case of $A(q)$ and $B(q)$, thus Eq. (5) is only a special case of Eq. (6). The Taylor series of $A(q)$ and $B(q)$ read

$$A(q) = \sum_{m=1}^{+\infty} \alpha_m q^m, \quad B(q) = \sum_{m=1}^{+\infty} \beta_m q^m,$$

which are assumed to be convergent at $q = 1$, i.e.

$$\sum_{m=1}^{+\infty} \alpha_m = \sum_{m=1}^{+\infty} \beta_m = 1.$$

Define the two vectors

$$\vec{\alpha} = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}, \quad \vec{\beta} = \{\beta_1, \beta_2, \beta_3, \ldots\}.$$

Now, the solution $\phi(\mathbf{r};q)$ of Eq. (6) is not only dependent upon the convergence-control parameter $\hbar$ but also the two vectors $\vec{\alpha}$ and $\vec{\beta}$. Thus, the generalized zeroth-order deformation equation (6) provides us greater freedom, or in other words, more possibility, to ensure the convergence of the Taylor series (2) at $q = 1$. This is the reason why $\vec{\alpha}$ and $\vec{\beta}$ are called the convergence-control vectors [6].

So, unlike perturbation techniques or other nonperturbation techniques, the HAM provides a convenient way to control the convergence of the series solutions by means of the so-called convergence-control parameter $\hbar$ and the convergence-control vectors $\vec{\alpha}$ and $\vec{\beta}$. Traditionally, the convergence-control parameter $\hbar$ is determined by plotting the so-called $\hbar$-curve, as suggested by Liao [2–4]. The main idea is to draw a curve of a certain quantity (mostly with physical meanings) versus $\hbar$, from which an interval of $\hbar$ which guarantees the convergence of the solution is identified. Because the convergence-control parameter $\hbar$ is an auxiliary parameter which has no physical meanings, all of the convergence series given by the HAM with different possible values of $\hbar$ tend to the same solution of a given equation, as proved by Liao [4] in general. Obviously, all of these possible values of $\hbar$ construct a set $\mathbf{R}_\hbar$ for the convergence-control parameters, and using any $\hbar \in \mathbf{R}_\hbar$ one can get a convergent series solution. However, the convergence rate is also dependent upon $\hbar$ but the so-called $\hbar$-curve approach can not give the "optimal" value of $\hbar$ in $\mathbf{R}_\hbar$.

In 2007, Yabushita et al. [10] suggested an "optimization method" for convergence-control parameters. Their approach is based on the square residual error

$$\Delta(\hbar) = \int_\Omega \left(\mathscr{N}\left[\sum_{k=0}^{M} u_k(\mathbf{r}) \right] \right)^2 \mathrm{d}\Omega$$

of a nonlinear equation $\mathscr{N}[u(\mathbf{r})] = 0$, where $\sum_{k=0}^{M} u_k(\mathbf{r})$ gives the Mth-order HAM approximation. Obviously, $\Delta(\hbar) \to 0$ (as $M \to +\infty$) corresponds to a convergent series solution. For given order M of approximation, the optimal value of $\hbar$ is given by a nonlinear algebraic equation

$$\frac{\mathrm{d}\Delta(\hbar)}{\mathrm{d}\hbar} = 0. \tag{8}$$

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In case of two coupled nonlinear equations (as seen in [10])

$$\mathscr{N}_1[u, v] = 0, \quad \mathscr{N}_2[u, v] = 0,$$

there may exist two convergence-control parameters $\hbar_1$ and $\hbar_2$. In this case, one can define the square residual error

$$\Delta(\hbar_1, \hbar_2) = \sum_{j=1}^{2} \int_\Omega \left(\mathscr{N}_j \left[\sum_{k=0}^{M} u_k, \sum_{k=0}^{M} v_k \right] \right)^2 d\Omega$$

and then gets the optimal $\hbar_1$ and $\hbar_2$ by solving two nonlinear algebraic equations

$$\frac{\partial \Delta(\hbar_1, \hbar_2)}{\partial \hbar_1} = 0, \quad \frac{\partial \Delta(\hbar_1, \hbar_2)}{\partial \hbar_2} = 0, \tag{9}$$

as shown by Yabushita et al. [10]. It is straightforward to use this approach to n ($n \geqslant 1$) coupled nonlinear differential equations.

Although the general zeroth-order deformation Eq. (6) was published 10 years ago and even more generalized form was reported [6], most users of the HAM applied Eq. (5), mainly due to its simplicity. Currently, Wu and Chueng [11] employed a special case $A(q) = q$ and $B(q) = \omega q + (1-\omega)q^2$, and determined the optimal values of the convergence-control parameters $\hbar$ and ω by considering the contour lines of a physical quantity versus $\hbar$ and ω. Wu and Chueng's optimization approach is in principle the same as that suggested by Yabushita et al. [10]. Currently, Wu and Chueng developed a two-parameter iterative approach based on the HAM [12]. Unlike Yabushita et al. [10] who used Eq. (5), Marinca et al. [13,14] employed such a homotopy equation

$$[(1-q)\mathscr{L}[\phi(\mathbf{r}; q) - u_0(\mathbf{r})] = H(q)\mathscr{N}[\phi(\mathbf{r}; q)], \tag{10}$$

where

$$H(q) = \sum_{k=1}^{+\infty} h_k q^k$$

is a convergent power series but $H(1) = 1$ is unnecessary. Obviously, Eq. (10) is a special case of Eq. (6) with the relationship $B(q) = q$ and $H(q) = \hbar A(q)$, i.e. $h_k = \hbar \alpha_k$. Like Yabushita et al. [10], Marinca et al. [13,14] determined the optimal values of the convergence-control parameters $h_1, h_2, h_3, \ldots, h_M$ by minimizing the square residual error $\Delta(h_1, h_2, h_3, \ldots, h_M)$ at a given order M of approximation in such a way

$$\frac{\partial \Delta(h_1, h_2, h_3, \ldots, h_M)}{\partial h_k} = 0, \quad 1 \leqslant k \leqslant M, \tag{11}$$

which leads to a set of M coupled nonlinear algebraic equations. Note that, although the above approach is named the "optimal homotopy asymptotic method", it is in principle in the frame of the homotopy analysis method.

Marinca's approach [13,14] is an important improvement of the HAM. However, using Marinca's approach [13,14], one had to solve a set of coupled nonlinear algebraic equations for the unknown convergence-control parameters $h_1, h_2, h_3, \ldots, h_M$. Obviously, if the order M of approximation is low, it is not difficult to solve such a set of nonlinear equations. However, as M increases, it becomes more and more difficult to solve it, and besides the necessary CPU time increases exponentially, as shown later in this paper.

To overcome this disadvantage of Marinca's approach [13,14], we present here an approach to determine the convergence-control parameters, which is efficient even for high-order M of approximation. Different from Marinca's approach [13,14] which solves a set of M coupled nonlinear algebraic equations for M unknown convergence-control parameters $h_1, h_2, h_3, \ldots, h_M$ as a whole, we minimize the square residual error of governing equations at each order so as to determine the optimal convergence-control parameters one by one. In this way, we need solve only one nonlinear algebraic equation about the unknown convergence-control parameter h_m at the mth-order of approximation, where $m = 1, 2, 3$ and so on, and therefore it is easy to get high-order approximations with much less CPU time, as mentioned later.

The basic idea of the present approach is described in Section 2. In Section 3, three examples are employed to illustrate the convergence, accuracy and computational efficiency of this approach, compared with Marinca's approach [13,14] and the traditional HAM as well. Conclusions and some discussions are given in the last section.

2. One-step optimal HAM

Consider the following equation:

$$\mathscr{N}[u(\mathbf{r})] = 0, \tag{12}$$

where $\mathscr{N}$ is a differential operator and $u(\mathbf{r})$ is the unknown function of the independent variable(s) $\mathbf{r} = \{r_1, r_2, r_3, \ldots\}$. For the sake of simplicity, we omit the corresponding boundary conditions, which can be handled in the similar way. Like Marinca's approach [13,14], we set $H(q) = \hbar A(q)$ and $B(q) = q$ in Liao's Eq. (6) to construct the zeroth-order deformation equation

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$$(1-q)\mathscr{L}[\phi(\mathbf{r};q)-u_0(\mathbf{r})] = H(q)\mathscr{N}[\phi(\mathbf{r};q)], \tag{13}$$

where $\mathscr{L}$ is an auxiliary linear operator, $u_0(\mathbf{r})$ an initial approximation of $u(\mathbf{r})$, $q \in [0,1]$ is the embedding parameter, and $H(q)$ is called the convergence-control function satisfying $H(0)=0$ and $H(1)\neq 0$.

From Eq. (13), we have

$$\phi(\mathbf{r};0) = u_0(\mathbf{r}), \quad \phi(\mathbf{r};1) = u(\mathbf{r}) \tag{14}$$

when $q=0$ and $q=1$, respectively. Thus, as the embedding parameter q increases from 0 to 1, $\phi(\mathbf{r};q)$ varies (or deforms) from the initial approximation $u_0(\mathbf{r})$ to the solution $u(\mathbf{r})$ of the original Eq. (12). Expand $\phi(\mathbf{r};q)$ and the convergence-control function $H(q)$ in Maclaurin's series as

$$\phi(\mathbf{r};q) = \sum_{m=0}^{+\infty} u_m(\mathbf{r}) q^m, \quad H(q) = \sum_{k=0}^{+\infty} h_k q^k, \tag{15}$$

where

$$u_m(\mathbf{r}) = \frac{1}{m!}\left.\frac{\partial^m \phi(\mathbf{r};q)}{\partial q^m}\right|_{q=0}, \quad h_k = \frac{1}{m!}\left.\frac{\partial^m H(q)}{\partial q^m}\right|_{q=0}. \tag{16}$$

Assuming that the two series in Eq. (15) are convergent at $q=1$, we have, due to Eq. (14), that

$$u(\mathbf{r}) = u_0(\mathbf{r}) + \sum_{m=1}^{+\infty} u_m(\mathbf{r}). \tag{17}$$

As suggested by Liao [3,4], differentiating the zeroth-order deformation equation (12) m times with respect to the embedding parameter q, then dividing it by $m!$ and finally setting $q=0$, we gain the so-called mth-order deformation equation for the unknown $u_m(\mathbf{r})$:

$$\mathscr{L}\left[u_m(\mathbf{r}) - \chi_m u_{m-1}(\mathbf{r})\right] = \sum_{k=1}^{m} h_k R_{m-k}(\mathbf{r}), \tag{18}$$

where

$$R_n(\mathbf{r}) = \frac{1}{n!}\left.\frac{\partial^n \mathscr{N}[\phi(\mathbf{r};q)]}{\partial q^n}\right|_{q=0}, \tag{19}$$

and

$$\chi_m = \begin{cases} 0, & \text{when } m \leqslant 1, \\ 1, & \text{when } m > 1. \end{cases} \tag{20}$$

For details, please refer to Liao [4].

The nth-order approximation of the solution $u(\mathbf{r})$ can be expressed as

$$\tilde{u}_n(\mathbf{r}) = u_0(\mathbf{r}) + \sum_{k=1}^{n} u_k(\mathbf{r}), \tag{21}$$

which is mathematically dependent upon the convergence-control parameter vector $\mathbf{h}_n = \{h_1, h_2, \ldots, h_n\}$. Let

$$\Delta_n(\mathbf{h}_n) = \int_\Omega \left(\mathscr{N}[\tilde{u}_n(\mathbf{r})]\right)^2 \mathrm{d}\Omega$$

denote the square residual error of the governing equation (12) at the nth-order of approximation, where $n = 1, 2, 3$ and so on. At the 1st-order of approximation, the square residual error Δ_1 is only dependent upon h_1 and thus we can gain the "optimal" value of h_1 by solving the nonlinear algebraic equation

$$\frac{\mathrm{d}\Delta_1}{\mathrm{d}h_1} = 0.$$

At the 2nd-order of approximation, the square residual error $\Delta_2(h_1, h_2)$ is a function of both of h_1 and h_2. Because h_1 is known, we can gain the "optimal" value of h_2 by solving one nonlinear algebraic equation

$$\frac{\mathrm{d}\Delta_2}{\mathrm{d}h_2} = 0.$$

Similarly, at the nth-order of approximation, the square residual error Δ_n contains only one unknown convergence-control parameter h_n, whose "optimal" value is determined by a nonlinear algebraic equation

$$\frac{\mathrm{d}\Delta_n}{\mathrm{d}h_n} = 0, \quad n \geqslant 1.$$

In this way, we can get accurate results at rather high-order of approximation. It may be expected that our one-step optimal HAM approach is computationally more efficient than Marinca's approach, since we need solve a nonlinear algebraic equation with only one unknown convergence-control parameter.

Noth that in previous applications of the HAM, $H(q) = \hbar q$ is mostly used, and $\hbar$ is chosen by plotting the so-called $\hbar$-curves. In this case, the square residual error Δ_n at a given order n of approximation is a function of $\hbar$, and the "optimal" value of the convergence-control parameter $\hbar$ is given by

$$\frac{d\Delta_n}{d\hbar} = 0.$$

This traditional approach is compared with our approach in the following section.

3. Some examples

In this section, three differential equations are employed to illustrate the validity of our one-step optimal HAM approach described in Section 2. The convergence, accuracy and efficiency of this optimization approach are investigated by comparing it with the traditional HAM approach and Marinca's approach [13,14]. The code is developed using symbolic computation software MATHEMATICA and the calculations are implemented on a PC with 2 GB RAM and 3 GHz CPU.

3.1. Example 1

First, consider a linear differential equation [15]:

$$u''(t) + tu'(t) - u(t) = f(t), \quad t \in [-1, 1], \tag{22}$$

subject to the boundary conditions

$$u(-1) = \exp(-5) + \sin(1), \quad u(1) = \exp(5) + \sin(1), \tag{23}$$

where

$$f(t) = (24 + 5t)\exp(5t) + (2 + 2t^2)\cos t^2 - (4t^2 + 1)\sin t^2. \tag{24}$$

The exact solution of (22) and (23) reads

$$u(t) = \exp(5t) + \sin(t^2). \tag{25}$$

Since $u(t)$ is defined in a finite domain $[-1,1]$, the solution of $u(t)$ can be expressed by the power series in t. Then, we can choose the auxiliary linear operator $\mathscr{L}u(t) = u''(t)$ and the initial approximation

$$u_0(t) = \frac{\exp(5) + \exp(-5)}{2} + \left[\frac{\exp(5) - \exp(-5)}{2}\right] t + \sin(1). \tag{26}$$

The right-hand side of Eq. (22) contains exponential and trigonometric functions which are not in the form of power series in t. So, we first expand it into Chebyshev's orthogonal polynomials as

$$f(t) \approx \frac{C_0}{2} + \sum_{k=1}^{K} C_k T_k(t), \tag{27}$$

where K is the order of the series, and

$$C_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(t)T_k(t)}{\sqrt{1-t^2}}\, dx, \quad k = 0, 1, \ldots, \tag{28}$$

$$T_k(t) = \cos(k \arccos(t)), \quad |t| \leqslant 1. \tag{29}$$

The 15th order Chebyshev's polynomial expansion of $f(t)$ is given by

$$\begin{aligned} f(t) \approx\ & 26 + 125t + 326.002t^2 + 562.502t^3 + 724.133t^4 + 755.187t^5 + 650.478t^6 + 480.665t^7 + 309.723t^8 \\ & + 177.175t^9 + 93.7633t^{10} + 43.6294t^{11} + 15.8078t^{12} + 6.40274t^{13} + 4.1018t^{14} + 1.36634t^{15}. \end{aligned} \tag{30}$$

The Mth-order approximation is given by

$$\tilde{u}_M(t) \approx u_0(t) + \sum_{k=1}^{M} u_k(t),$$

where $u_m(t)$ is determined by the mth-order deformation equation

$$\mathscr{L}[u_m(t) - \chi_m u_{m-1}(t)] = \sum_{k=1}^{m} h_k R_{m-k}(t), \tag{31}$$

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Table 1
Comparison of Δ_n and CPU time (seconds) given by one-step optimal HAM and the traditional HAM for Example 1.

One-step optimal HAM				Traditional HAM		
Order n	h_n	Δ_n	CPU (s)	$\hbar$	Δ_n	CPU (s)
1	−8.2515E−1	12242	0.125	−0.82515	12242	0.291
2	−1.8218E−2	884.6	0.266	−0.77697	1250	0.359
3	4.1013E−3	5.58	0.421	−0.75933	9.94	0.827
4	−1.1324E−4	1.12	0.561	−0.75117	1.01	1.264
5	−1.9777E−4	2.28E−1	0.765	−0.74752	1.43E−2	2.131
6	6.7750E−5	2.75E−3	0.905	−0.75094	1.10E−3	3.219
7	−2.6473E−6	8.51E−4	1.061	−0.75089	5.93E−5	4.536
8	−5.6510E−6	2.48E−4	1.233	−0.74128	1.84E−6	5.832
9	2.0488E−6	2.48E−6	1.513	−0.70904	4.24E−7	7.304

subject to boundary conditions

$$u_m(-1) = 0, \quad u_m(1) = 0, \tag{32}$$

where

$$R_k(t) = u_k''(t) + t u_k'(t) - u_k(t) - (1 - \chi_{k+1}) f(t). \tag{33}$$

The square residual error at the Mth-order of approximation is defined by

$$\Delta_M = \int_{-1}^{1} [\tilde{u}_M''(t) + t\tilde{u}_M'(t) - \tilde{u}_M(t) - f(t)]^2 \, dt. \tag{34}$$

It is found that

$$\Delta_1 = 1.9259 \times 10^6 + 4.6384 \times 10^6 h_1 + 2.81066 \times 10^6 h_1^2, \tag{35}$$

$$\begin{aligned} \Delta_2 = {} & 1.9259 \times 10^6 + 9.2769 \times 10^6 h_1 + 1.6943 \times 10^7 h_1^2 + 1.3925 \times 10^7 h_1^3 + 4.3537 \times 10^6 h_1^4 \\ & + 4.6384 \times 10^6 h_2 + 1.1242 \times 10^7 h_1 h_2 + 6.9628 \times 10^6 h_1^2 h_2 + 2.8106 \times 10^6 h_2^2, \end{aligned} \tag{36}$$

and so on. Note that Δ_n contains n convergence-control parameters $h_1, h_2, h_3, \ldots, h_n$. Our approach gives the "optimal" value of the first convergence-control parameter h_1 by solving the equation $d\Delta_1/dh_1 = 0$, which leads to $h_1 = -0.82515$ with the corresponding minimum square residual error $\Delta_1 = 12241$. Then, Δ_2 is only dependent upon h_2 since h_1 is regarded as known. Thus, we obtain the "optimal" value of h_2 by solving the algebraic equation $d\Delta_2/dh_2 = 0$, which gives $h_2 = -0.018218$ with the corresponding minimum square residual error $\Delta_2 = 884.64$. In this way, we gain the "optimal" value of the convergence-control parameters $h_1, h_2, h_3, \ldots$, one by one, until the accurate enough approximation is obtained.

The comparison of the traditional HAM and the one-step optimal approach is given in Table 1. For both approaches, the square residual error Δ_n decreases quickly as the order of approximation increases. However, the one-step optimal approach needs less CPU time than the traditional HAM approach which first regards the unique convergence-control parameter $\hbar$ as unknown and then gives its optimal value by minimizing the square residual error. This is mainly because, containing a unknown convergence-control parameter $\hbar$, the expression of $u_m(t)$ becomes more and more complicated as m increases, and thus more and more CPU time is needed to solve the corresponding high-order deformation equations. Thus, since Marinca's optimization approach [13,14] contains more unknown convergence-control parameters, it needs even more CPU times, especially for high-order approximations, as shown in Table 2 and Fig. 1. For example, at the 8th-order of approximation, Marinca's approach needs 1194 times more CPU time than the traditional HAM, and even 5651 times more CPU time than the one-step optimal approach, as shown in Table 3.

Table 2
h_M, Δ_M and CPU time (seconds) at different order M of approximations given by Marinca's approach for Example 1.

M	1	2	3	4	5	6
h_1	−0.8252	−0.7359	−0.8048	−0.8051	−0.8436	−0.8436
h_2	/	−2.415E−2	−1.810E−2	−1.142E−2	−1.572E−2	−1.572E−2
h_3	/	/	3.371E−3	1.925E−3	1.739E−3	1.720E−3
h_4	/	/	/	−1.921E−4	−3.003E−4	−2.401E−4
h_5	/	/	/	/	2.942E−5	−9.932E−5
h_6	/	/	/	/	/	9.7917E−6
Δ_M	1.22×10^4	78.6	0.224	6.82E−03	8.49E−05	2.91E−05
CPU	0.144	0.608	2.17	9.19	37.5	245.2

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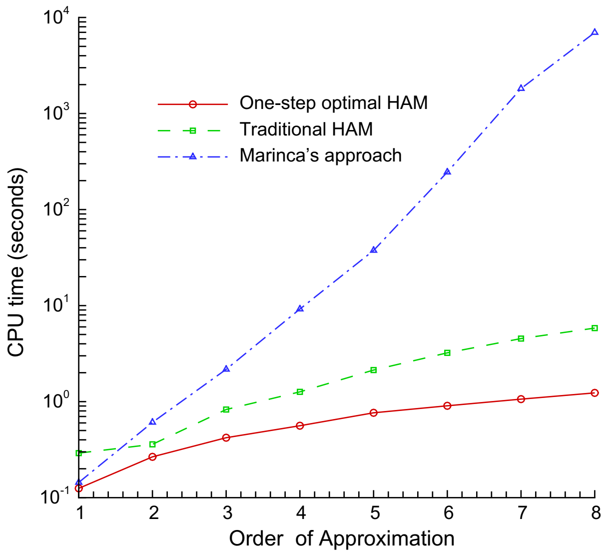

Fig. 1. Comparison of CPU time of different approaches for Example 1.

Table 3
Comparison of CPU time (seconds) of different approaches for Example 1.

Order M	One-step optimal HAM	Traditional HAM	Marinca's approach
1	0.125	0.291	0.144
2	0.266	0.359	0.608
3	0.421	0.827	2.17
4	0.561	1.264	9.19
5	0.765	2.131	37.5
6	0.905	3.219	245.2
7	1.061	4.536	1817.5
8	1.233	5.832	6969.5

Note that, at high-order approximations, the square residual error given by the traditional optimization approach decays more quickly than our one-step optimal approach, as shown in Table 1. And square residual errors given by Marinca's approach decays even faster than the traditional optimal HAM approach, as shown in Table 2. Note that, the square residual error at the 9th-order approximation given by one-step optimal approach with only 1.513 s CPU time is less than that at the 6th-order approximation by Marinca's approach with 245.2 s CPU time. Thus, our one-step optimal approach is much more efficient than Marinca's approach.

3.2. Example 2

Then, let us consider a nonlinear differential equation [16]:

$$u''(t) - u(t)u''(t) - \frac{u'(t)^2}{2} - \frac{1}{2} = 0, \tag{37}$$

subject to boundary conditions

$$u(0) = 0, \quad u(1) = -1/2. \tag{38}$$

Similarly, we choose the initial approximation

$$u_0(t) = -\frac{t}{2}, \tag{39}$$

which satisfies the boundary conditions (38), and the same auxiliary linear operator as that in Example 1. The mth-order deformation equation reads

$$\mathscr{L}[u_m(t) - \chi_m u_{m-1}(t)] = \sum_{k=1}^{m} h_k R_{m-k}(t), \tag{40}$$

Table 4
Comparison of Δ_n and CPU time (seconds) given by one-step optimal HAM and the traditional HAM for Example 2.

One-step optimal HAM				Traditional HAM		
Order n	$\hbar_n$	Δ_n	CPU	$\hbar$	Δ_n	CPU
1	−7.4849E-1	1.94E-2	0.078	−0.74849	1.94E-2	0.125
2	−1.2724E-3	1.75E-3	0.187	−0.77608	1.66E-3	0.234
3	−3.7716E-4	1.25E-4	0.328	−0.78472	9.69E-5	0.500
4	−4.8188E-4	8.57E-6	0.375	−0.78695	5.89E-6	0.515
5	−9.7056E-5	5.72E-7	0.516	−0.78566	3.70E-7	1.203
6	−1.9365E-6	3.84E-8	0.625	−0.78131	2.85E-8	1.234
7	1.8106E-6	2.72E-9	0.796	−0.77855	2.15E-9	1.818
8	2.7190E-7	2.01E-10	0.906	−0.77205	1.75E-10	2.640
9	5.3089E-8	1.52E-11	1.125	−0.76910	1.44E-11	3.765
10	2.1653E-8	1.17E-12	1.281	−0.76386	1.20E-12	5.219

subject to boundary conditions

$$u_m(0) = 0, \quad u_m(1) = 0, \tag{41}$$

where

$$R_n(t) = u_n''(t) - \sum_{j=0}^{n} \left[u_j(t)u_{n-j}''(t) - \frac{1}{2}u_j'(t)u_{n-j}'(t) \right] - \frac{1}{2}(1 - \chi_{n+1}). \tag{42}$$

Similarly, it is found that the square residual error given by our one-step optimal approach decays quickly as the order of approximation increases, as shown in Table 4. Besides, less CPU time is used than the traditional HAM approach. Note that Marinca's approach needs much more CPU times, as shown in Tables 5 and 6, and also Fig. 2.

This example indicates that our one-step optimal approach is valid for nonlinear equations and is much more efficient than Marinca's approach.

3.3. Example 3

Consider the following nonlinear differential equation [16]:

$$u''(t) - u''(t)u(t)^2 - u(t)u'(t)^2 = 0, \tag{43}$$

subject to the boundary conditions

$$u(0) = 0, \quad u(1) = \alpha, \tag{44}$$

Table 5
$\hbar_M, \Delta_M$ and CPU time (seconds) at different order M of approximation given by Marinca's approach for Example 2.

M	1	2	3	4	5	6
$\hbar_1$	−0.7485	−0.7771	−0.7854	−0.7868	−0.7873	−0.7873
$\hbar_2$	/	−2.4095E-3	−3.2858E-3	1.7860E-3	3.6330E-3	5.1430E-3
$\hbar_3$	/	/	−4.5908E-4	−7.0998E-4	-5.4369E-4	5.1127E-4
$\hbar_4$	/	/	/	−6.2957E-5	−5.4645E-4	5.0127E-4
$\hbar_5$	/	/	/	/	−1.3894E-5	−7.9943E-6
$\hbar_6$	/	/	/	/	/	−5.4226E-7
Δ_M	1.95E-2	1.66E-3	9.56E-5	5.55E-6	3.08E-7	1.6931E-8
CPU	0.044	0.308	4.38	23.1	446.3	7865.09

Table 6
Comparison of CPU time (seconds) of different approaches for Example 2.

Order M	One-step optimal HAM	Traditional HAM	Marinca's approach
1	0.078	0.125	0.044
2	0.187	0.234	0.308
3	0.328	0.500	4.38
4	0.375	0.515	23.1
5	0.516	1.203	446.3
6	0.625	1.234	7865.09
7	0.796	1.818	69502.2

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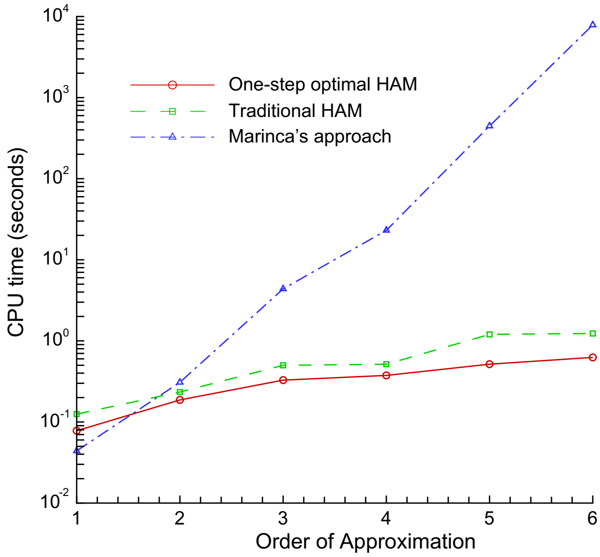

Fig. 2. Comparison of CPU time of different approaches for Example 2.

whose exact solution is expressed in the implicit form

$$\lambda t = u\sqrt{1-u^2} + \arcsin u, \tag{45}$$

where $\lambda = \alpha\sqrt{1-\alpha^2} + \arcsin\alpha$.

We choose the initial approximation

$$u_0(t) = \alpha t \tag{46}$$

and the same auxiliary linear operator as that in Example 1. The high-order deformation equation reads

$$\mathscr{L}[u_m(t) - \chi_m u_{m-1}(t)] = \sum_{k=1}^{m} h_k R_{m-k}(t), \tag{47}$$

subject to boundary conditions

$$u_m(0) = 0, \quad u_m(1) = 0, \tag{48}$$

where

$$R_n(t) = u_n'' - \sum_{j=0}^{n} u_{n-j}'' \sum_{k=0}^{j} u_k u_{j-k} - \sum_{j=0}^{n} u_{n-j} \sum_{k=0}^{j} u_k' u_{j-k}'. \tag{49}$$

In case of $\alpha = 1/2$, the square residual error given by our one-step optimal approach decays quickly the order of approximation increases, as shown in Table 7. Besides, less CPU time is needed than the traditional HAM approach. Furthermore, Marinca's approach needs much more CPU times than the traditional HAM approach and especially than one-step optimal

Table 7
Comparison of Δ_n and CPU time (seconds) given by one-step optimal HAM and the traditional HAM for Example 3 in case of $\alpha = 1/2$.

One-step optimal HAM				Traditional HAM		
Order n	h_n	Δ_n	CPU	h	Δ_n	CPU
1	−1.2072	1.62E-4	0.234	−1.2072	1.62E-4	0.171
2	−7.5878E-3	3.31E-6	0.406	−1.1837	3.23E-6	0.485
3	−1.6369E-4	8.24E-8	0.719	−1.1713	6.71E-8	1.016
4	−1.1022E-5	2.22E-9	1.031	−1.1648	1.38E-9	1.985
5	−1.9823E-7	6.06E-11	1.360	−1.1608	2.89E-11	3.515
6	−1.2596E-7	1.64E-12	1.781	−1.1581	6.03E-13	6.047
7	9.9048E-9	4.46E-14	2.172	−1.1561	1.26E-14	9.531
8	5.2092E-11	1.20E-15	2.640	−1.1546	2.65E-16	15.578
9	7.4391E-13	3.22E-17	3.156	−1.1534	5.57E-18	19.750
10	−9.6024E-12	8.61E-19	3.672	−1.1525	1.16E-19	34.641

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Table 8
h_M, Δ_M and CPU time (seconds) at different order of approximation M given by Marinca's approach for Example 3 in case of $\alpha = 1/2$.

M	1	2	3	4	5
h_1	−1.2072	−1.1821	−1.1681	−1.1607	−1.1607
h_2	/	−7.8045E−3	−6.0729E−3	−5.1929E−3	−5.1929E−3
h_3	/	/	−1.7898E−4	−1.4808E−4	−1.4812E−4
h_4	/	/	/	−1.0919E−5	−1.0909E−5
h_5	/	/	/	/	0
Δ_M	1.62E−4	3.02E−6	5.01E−8	7.88E−10	1.2830E−11
CPU	0.112	3.50	14.9	476.7	32086.4

approach, as shown in Tables 8 and 9, and also Fig. 3. However, in case of $\alpha = 9/10$, the square residual error given by the one-step optimal approach first decays, although slowly, but then increases, as shown in Table 10. In contrast, the square residual errors given by the traditional HAM and Marinca's approach decay monotonously. This illustrates the weakness of the one-step optimal approach.

4. Conclusion and discussion

Marinca et al. [13,14] proposed an approach to find the optimal convergence-control parameters in the frame of the HAM. At a given order M of approximation, it is necessary to solve a set of coupled nonlinear algebraic equations with the M unknown convergence-control parameters $h_1, h_2, h_2, \ldots, h_M$ by means of Marinca's approach. So, Marinca's approach is time-consuming, especially for large M, as shown in Figs. 1–3. Even worse, if an approximation at a given order M is not accurate enough, a higher-order approximation is necessary and then a completely new set of coupled algebraic equations with more unknowns must be solved. So, Marinca's approach [13,14] is not efficient in practice.

To overcome this disadvantage of Marinca's approach [13,14], we present one-step optimization approach for the convergence-control parameters of the HAM. Unlike Marinca's approach [13,14], only one algebraic equation is solved at each order

Table 9
Comparison of CPU time (seconds) of different approaches for Example 3 in case of $\alpha = 1/2$.

Order M	One-step optimal HAM	Traditional HAM	Marinca's approach
1	0.234	0.171	0.112
2	0.406	0.485	3.50
3	0.719	1.016	14.9
4	1.031	1.985	476.7
5	1.360	3.515	32086.4

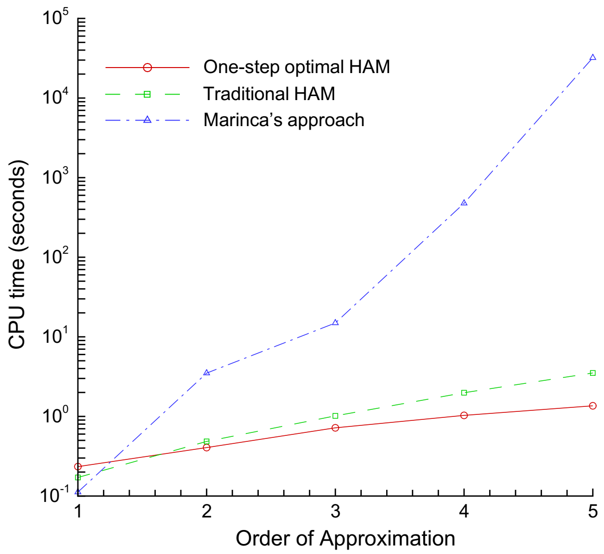

Fig. 3. Comparison of CPU time of different approaches for Example 3.

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Table 10
Comparison of Δ_n given by one-step optimal HAM and the traditional HAM for Example 3 in case of $\alpha = 9/10$.

One-step optimal HAM			Traditional HAM		
Order n	h_n	Δ_n	Order n	$\hbar$	Δ_n
1	-1.0133	1.10E-1	1	-1.0133	1.10E-1
5	6.9401E-2	4.42E-2	5	-1.7029	8.89E-3
10	2.4243E-2	2.88E-2	10	-1.7046	2.31E-4
15	1.2823E-2	2.29E-2	15	-1.6997	5.72E-6
20	9.0659E-3	1.0095	20	-1.6963	1.36E-7

of approximation, until accurate enough approximation is obtained. This approach needs much less CPU time and thus is more efficient in practice, as shown in Figs. 1–3. Besides, our one-step optimal approach automatically determines the convergence-control parameters one by one, and therefore it is easy to apply this approach to develop symbolic computation codes for some types of nonlinear differential equations. However, it is a pity that sometimes one can not get convergent result by means of this one-step optimal approach. This disadvantage can be overcome by using multiple-step optimal approach (like Marinca's approach [13,14]) first, and then one-step optimal approach followed. More investigations are needed to obtain optimal values of the convergence-control parameters in the frame of the HAM, especially when the even more generalized zeroth-order deformation equations mentioned in [6] is considered.

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