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On the analytic solutions of the nonhomogeneous Blasius problem

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Abstract

In this article a totally analytic solution of the nonhomogeneous Blasius problem is obtained using the homotopy analysis method (HAM). This solution converges for $0 \leqslant \eta < \infty$. Existence and nonuniqueness of solution is also discussed. An implicit relation between the velocity at the wall λ and the shear stress $\alpha = f''(0)$ is obtained. The results presented here indicate that two solutions exist in the range $0 < \lambda < \lambda_c$, for some critical value λ_c one solution exists for $\lambda = \lambda_c$, and no solution exists for $\lambda > \lambda_c$. An analytical value of the critical value of λ_c was also obtained for the first time.

MSC: 65-xx

Keywords: Blasius problem; Analytic solution; Homotopy analysis method

1. Introduction

The standard nonhomogeneous Blasius equation is given by

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0 \tag{1}$$

with initial and boundary conditions

$$f(0) = 0, \quad f'(0) = -\lambda \quad \text{and} \quad f'(\infty) = 1. \tag{2}$$

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This equation is associated with the boundary layer flow over a moving plate with constant velocity λ. The function $f(\eta)$ is the nondimensional stream function and η is the similarity ordinate. The derivation of this equation from the classical Navier–Stokes equations can be found in [7,9,21].

The existence and nonuniqueness of the solution of this problem were discussed by [6,7,12,23], and numerical techniques were employed to solve Eq. (1) with the following initial conditions:

$$f(0) = 0, \quad f'(0) = -\lambda, \quad f''(0) = \alpha. \tag{3}$$

The shooting method was employed to solve the above initial value problem and the value of $f'(\eta)$ as $\eta \to \infty$ was observed while the parameter α was changed. The solution of the boundary value problem described by Eqs. (1) and (2) exists if there exists an α such that $f'(\eta) \to 1$ as $\eta \to \infty$. The numerical results presented in [7] indicate that two solutions exist in the range $0 < \lambda < \lambda_c$, where λ_c was found to be $0.3546\ldots$, one solution exists for $\lambda = \lambda_c$, and no solution exists for $\lambda > \lambda_c$.

In this article, the question of existence and uniqueness of an analytical solution will be addressed. The application of Adomian decomposition method will be discussed in the next section. Section 3 will address the derivation of the analytic solution using the homotopy analysis method (HAM). In Section 4 the question of existence and uniqueness will be discussed and conclusion remarks are presented in Section 5.

2. Analytical solution of the problem

Several attempts were made to derive an analytical solution of the problem for the case $\lambda = 0$, see [8–11,13–20]. One of these methods is the Adomian decomposition method (ADM). The ADM [1–5] is quantitative rather than qualitative, analytic, requiring neither linearization nor perturbation and continuous with no resort to discretization. It consists of splitting the given equation into linear and nonlinear parts inverting the highest-order derivative operator contained in the linear operator on both sides, and then identifying the initial and/or boundary conditions and the terms involving the independent variables alone as initial approximation. After that, we decompose the unknown function into a series whose components are to be determined, and then decompose the nonlinear function in terms of special polynomials that are called Adomian's polynomials. Finally, we find the successive terms of the series solution by a recurrent relation using Adomian polynomials.

Following the above procedures, the nonlinear differential equation

$$F(x, y(x)) = 0 \tag{4}$$

can be split into the two components

$$L(y(x)) + N(y(x)) = 0, \tag{5}$$

where L and N are the linear and nonlinear parts of F, respectively, and L is an invertible operator. Rewrite Eq. (5) as

$$L(y) = -N(y). \tag{6}$$

Because L is invertible, one can apply the inverse operator L^{-1} and gets the solution y of Eq. (4) as

$$y = -L^{-1}(N(y)) + \varphi(x), \tag{7}$$

where $\varphi(x)$ is the constant of integration and satisfies the condition $L(\varphi)=0$. Assume that the solution y can be represented as an infinite series of the form

$$y=\sum_{n=0}^{\infty} y_n \tag{8}$$

and the nonlinear term $N(y)$ is written as an infinite series of the form

$$N(y)=\Phi(y)=\sum_{n=0}^{\infty} A_n, \tag{9}$$

where the Adomian polynomials A_n of $\Phi(y)$ can be evaluated by the formula, see [1,3,22],

$$A_n=\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}p^n}\Phi\left(\sum_{n=0}^{\infty}(p^n y_n)\right)\Bigg|_{p=0}. \tag{10}$$

Now substituting Eqs. (8) and (9) in Eq. (7) gives

$$\sum_{n=0}^{\infty} y_n=\varphi(x)-L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{11}$$

Each term in Eq. (11) is given by the recurrent relation

$$\begin{aligned} &y_0=\varphi(x),\\ &y_{n+1}=-L^{-1}(A_n), \quad n\geqslant 0. \end{aligned} \tag{12}$$

For the problem under consideration, the classical splitting of the problem into linear and nonlinear parts is given by

$$\begin{aligned} &L(f)=f''',\\ &N(f)=\tfrac{1}{2}f''(\eta)f(\eta). \end{aligned} \tag{13}$$

The initial solution $f_0(\eta)$ will be the solution of the linear equation

$$L(f)=f'''(\eta)=0 \tag{14}$$

subject to the initial conditions

$$f(0)=0, \quad f'(0)=-\lambda, \quad f''(0)=\alpha \tag{15}$$

which will be given by

$$f_0(\eta)=\frac{\eta^2\alpha}{2}-\eta\lambda. \tag{16}$$

Assume that the solution $f(\eta)$ is given by the series solution as

$$f(\eta)=\sum_{n=0}^{\infty} f_n(\eta). \tag{17}$$

Then the nth term of the above series can be obtained from the recurrence relation

$$f_n(\eta)=-L^{-1}[A_{n-1}], \tag{18}$$

where $L^{-1}=\int\int\int \mathrm{d}\eta\,\mathrm{d}\eta\,\mathrm{d}\eta$ and the nth term of the Adomian polynomial A_n is given by

$$A_n=\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}p^n}N\left(\sum_{n=0}^{\infty}(p^n y_n)\right)\Bigg|_{p=0}. \tag{19}$$

The first few terms are given by

$$\begin{aligned} A_0&=\tfrac{1}{2}(f_0'' f_0),\\ A_1&=\tfrac{1}{2}(f_1'' f_0+f_0'' f_1),\\ A_2&=\tfrac{1}{2}(f_2'' f_0+f_1'' f_1+f_0'' f_2),\\ &\ \vdots \end{aligned} \tag{20}$$

Using symbolic computations to solve Eq. (18), we get the following:

$$\begin{aligned} f_0(\eta)&=\frac{\eta^2\alpha}{2}-\eta\lambda,\\ f_1(\eta)&=\frac{-1}{240}\eta^5\alpha^2+\frac{1}{48}\eta^4\alpha\lambda,\\ f_2(\eta)&=\frac{11\eta^8\alpha^3}{161280}-\frac{11\eta^7\alpha^2\lambda}{20160}+\frac{\eta^2\alpha\lambda^2}{960},\\ &\ \vdots \end{aligned}$$

Then the solution $f(\eta)$ will be

$$f(\eta)=f_0(\eta)+f_1(\eta)+f_2(\eta)+\cdots . \tag{21}$$

The above solution, given by Eq. (21), has the following two drawbacks.

(1) It is clear from the above terms that the solution is half numeric and half analytic because of the unknown value of α, which has to be known in advance to be able to get the analytical solution.
(2) For $\lambda=0$, the above series will be the series solution of the homogeneous Blasius problem which was addressed by many authors, see [21]. It was shown that the above series converges on the restricted domain $|\rho|<\rho_0\simeq 5.690$.

These two drawbacks initiate the need to search for a different approach to solve the problem under consideration, Eqs. (1) and (2). This approach is the HAM. For more details about the HAM, see [13–20].

3. Homotopy analysis method

Consider a nonlinear differential operator $\tilde{N}$, let $\hbar \neq 0$ and p be complex numbers, and $A(p)$ and $B(p)$ be two complex functions analytic in the region $|p| \leqslant 1$, which satisfy

$$A(0) = B(0) = 0, \quad A(1) = B(1) = 1, \tag{22}$$

respectively. Let

$$A(p) = \sum_{k=1}^{\infty} \alpha_{1,k} p^k, \quad B(p) = \sum_{k=1}^{\infty} \beta_{1,k} p^k \tag{23}$$

denote Maclaurin's series of $A(p)$ and $B(p)$, respectively.

The complex functions $A(p)$ and $B(p)$ are called the embedding functions and p is the embedding parameter.

Consider the nonlinear differential equation in general form

$$\tilde{N}(u(r)) = 0, \quad r \in \Omega, \tag{24}$$

where $\tilde{N}$ is a differential operator and $u(r)$ is a solution defined in the region $r \in \Omega$. Applying the HAM to solve it, we first need to construct the following family of equations:

$$[1 - B(p)]\{\pounds[\theta(r, p) - u_0(r)]\} = \hbar A(p)\tilde{N}[\theta(r, p)], \tag{25}$$

where £ is a properly selected auxiliary linear operator satisfying

$$\pounds(0) = 0, \tag{26}$$

$\hbar \neq 0$ is an auxiliary parameter, and $u_0(r)$ is an initial approximation. According to the definition of the embedding functions $A(p)$ and $B(p)$, Eq. (25) gives

$$\theta(r, 0) = u_0(r) \tag{27}$$

when $p = 0$. Similarly, when $p = 1$, Eq. (25) is the same as Eq. (24) so that we have

$$\theta(r, 1) = u(r). \tag{28}$$

Suppose that Eq. (24) has solution $\theta(r, p)$ that converges for all $0 \leqslant p \leqslant 1$ for properly selected $\hbar$, $A(p)$ and $B(p)$. Suppose further that $\theta(r, p)$ is infinitely differentiable at $p = 0$, that is

$$\theta_0^k(r) = \left.\frac{\partial^k \theta(r, p)}{\partial p^k}\right|_{p=0}, \quad k = 1, 2, 3, \ldots \, . \tag{29}$$

Thus as p increases from 0 to 1, the solution $\theta(r, p)$ of Eq. (25) varies continuously from the initial approximation $u_0(r)$ to the solution $u(r)$ of the original Eq. (24). Clearly, Eqs. (27) and (28) give an indirect relation between the initial approximation $u_0(r)$ and the general solution $u(r)$. The HAM depends on finding a direct relationship between the two solutions which can be described as follows.

Consider the Maclaurin's series of $\theta(r, p)$ about p:

$$\theta(r, p) = \theta(r, 0) + \sum_{k=1}^{\infty} \left(\left.\frac{\partial^k \theta(r, p)}{\partial p^k}\right|_{p=0} \right) \frac{p^k}{k!}. \tag{30}$$

Assume that the series above converges at $p=1$. From Eqs. (27) and (30), we have the relationship

$$u(r)=u_0(r)+\sum_{m=1}^{\infty}\phi_m(r), \tag{31}$$

where

$$\phi_m(r)=\frac{\theta_0^m(r)}{m!}, \quad m\geqslant 1. \tag{32}$$

To derive the governing equation of $\phi_m(r)$, we differentiate Eq. (25) m times with respect to p. We get

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\frac{\mathrm{d}^k[1-B(p)]}{\mathrm{d}p^k}\,\frac{\mathrm{d}^{m-k}}{\mathrm{d}p^{m-k}}\{£[\theta(r,p)]-£[u_0(r)]\}\\ =\hbar\sum_{k=0}^{m}\binom{m}{k}\frac{\mathrm{d}^kA(p)}{\mathrm{d}p^k}\,\frac{\mathrm{d}^{m-k}\tilde{N}[\theta(r,p)]}{\mathrm{d}p^{m-k}}.\end{aligned} \tag{33}$$

Divide Eq. (33) by $m!$ and then set $p=0$; we get the mth-order deformation equations

$$£\left[\phi_m(r)-\sum_{k=1}^{m-1}\beta_{1,k}\phi_{m-k}(r)\right]=R_m(r), \tag{34}$$

where

$$R_m(r)=\hbar\sum_{k=1}^{m}\alpha_{1,k}\delta_{m-k}(r) \tag{35}$$

and $\delta_k(r)$ are the HAM polynomials and they are given by

$$\delta_k(r)=\frac{1}{k!}\left.\frac{\mathrm{d}^k\tilde{N}[\theta(r,p)]}{\mathrm{d}p^k}\right|_{p=0}. \tag{36}$$

It is very important to emphasize that Eq. (32) is linear. If the first $(m-1)$th-order approximations have been obtained, the left-hand side $R_m(r)$ will be obtained. So, using the selected initial approximation $u_0(r)$, we can obtain $\phi_1(r)$, $\phi_2(r)$, $\phi_3(r), \ldots,$ one after the other in order. Therefore, according to Eq. (34), we convert the original nonlinear problem into an infinite sequence of linear sub-problems governed by Eq. (34).

To use the above techniques to solve the differential equation under consideration, we choose the two embedding parameters $A(p)$ and $B(p)$ as

$$\begin{aligned}A(p)&=p,\\ B(p)&=p\end{aligned} \tag{37}$$

and the auxiliary parameter $\hbar = -1$. The linear operator L is chosen to be

$$L(f) = f'''(\eta) + \beta f''(\eta), \tag{38}$$

where β is an auxiliary parameter. The nonlinear equation will be

$$\tilde{N}(f) = f'''(\eta) + \tfrac{1}{2} f''(\eta) f(\eta). \tag{39}$$

Then following the setting given by Eq. (25), one gets

$$(1 - p)L(f) = -p\tilde{N}(f).$$

This will lead to the new linear part of the problem given by the left side of the previous equation, and the solution $f_0(\eta)$ is now the solution of the linear equation

$$f'''(\eta) + \beta f''(\eta) = 0 \tag{40}$$

subject to the initial and boundary conditions given by Eq. (2). Then the initial approximation $f_0(\eta)$ will be

$$f_0(\eta) = -\left(\frac{\alpha}{\beta^2}\right) + \frac{\alpha}{e^{\eta\beta}\beta^2} - \frac{\eta(-\alpha + \beta\lambda)}{\beta}.$$

The recurrence relation for f_n will be

$$f'''_{n+1} + \beta f_{n+1} = \Phi_n \quad \text{for } n \geqslant 0, \tag{41}$$

where the HAM polynomials Φ_n now are given by the relation

$$\Phi_n = \frac{1}{n!} \frac{d^n}{dp^n} \tilde{N} \left(\sum_{n=0}^{\infty} (p^n y_n) \right) \Bigg|_{p=0}. \tag{42}$$

The first few terms of these polynomials are given by

$$\begin{aligned}
\Phi_0 &= \beta f_0''(\eta) - \tfrac{1}{2}(f_0'' f_0), \\
\Phi_1 &= \beta f_1''(\eta) - \tfrac{1}{2}(f_1'' f_0 + f_0'' f_1), \\
\Phi_2 &= \beta f_2''(\eta) - \tfrac{1}{2}(f_2'' f_0 + f_1'' f_1 + f_0'' f_2), \\
&\ \vdots
\end{aligned} \tag{43}$$

The linear system of differential equations given by Eq. (41) is solved for $f_n(\eta)$, $n = 1, 2, 3, \ldots$, subject to the homogeneous conditions

$$f_n(0) = 0, \quad f_n'(0) = 0, \quad f_n''(0) = 0. \tag{44}$$

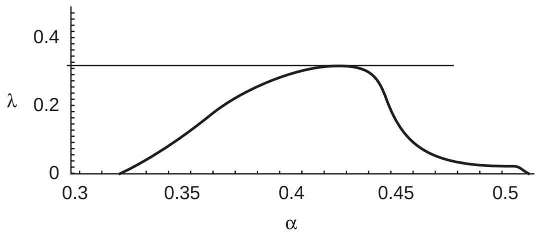

Fig. 1. Implicit curve of the function $f(\alpha, \lambda) = 1$.

With the help of symbolic computations performed by Mathematica, one can solve the above recurrence relation for $f_n(\eta)$. The first few terms of the solution $f_n(\eta)$ are given as

$$\begin{aligned}
f_0(\eta) &= -\left(\frac{\alpha}{\beta^2}\right) + \frac{\alpha}{e^{\eta\,\beta}\,\beta^2} - \frac{\eta\,(-\alpha + \beta\,\lambda)}{\beta},\\
f_1(\eta) &= \frac{0.125\alpha^2}{e^{2\eta\beta}\beta^5} - \frac{1\alpha^2}{e^{1\eta\beta}\beta^5} - \frac{0.5\eta\alpha^2}{e^{1\eta\beta}\beta^4} - \frac{0.25\eta^2\alpha^2}{e^{1\eta\beta}\beta^3} + \frac{2\alpha}{e^{1\eta\beta}\beta^2} + \frac{\eta\alpha}{e^{1\eta\beta}\beta}\\
&\quad + \frac{1.5\alpha\lambda}{e^{1\eta\beta}\beta^4} + \frac{\eta\alpha\lambda}{e^{1\eta\beta}\beta^3} + \frac{0.25\eta^2\alpha\lambda}{e^{1\eta\beta}\beta^2} - \frac{0.25\eta\left(\alpha^2 - 4\alpha\beta^3 - 2\alpha\beta\lambda\right)}{\beta^4}\\
&\quad - \frac{0.125\left(-7\alpha^2 + 16\alpha\beta^3 + 12\alpha\beta\lambda\right)}{\beta^5},\\
f_2(\eta) &= \frac{0.5\eta^2\alpha}{e^{1\eta\beta}} + \frac{0.0173611\alpha^3}{e^{3\eta\beta}\beta^8} - \frac{0.46875\alpha^3}{e^{2\eta\beta}\beta^8} + \frac{1.25\alpha^3}{e^{1\eta\beta}\beta^8} - \frac{0.1875\eta\alpha^3}{e^{2\eta\beta}\beta^7} + \frac{1.3125\eta\alpha^3}{e^{1\eta\beta}\beta^7}\cdots,\\
&\ \vdots
\end{aligned} \tag{45}$$

In [21] it has been shown that for $\lambda = 0$, the above solution converges in the whole domain of definition of f, namely for $0 \leqslant \eta < \infty$. Experimenting with several values of β, we realize that the above solution converges for $\beta > 1$. The value of β chosen for the results presented here is $\beta \simeq 2.1$.

3.1. Nonuniqueness of solution

As mentioned earlier, the nonuniqueness of solution of this problem was discussed by many authors. Allan and Abu Saris [7] employed numerical techniques to show the nonuniqueness of solution for $0 < \lambda < \lambda_c$, where λ_c was found to be 0.3546—one solution exists for $\lambda = \lambda_c$ and no solution exists for $\lambda > \lambda_c$. With the help of the analytical solution obtained in the previous section, Eq. (46), one can derive an implicit relation between the initial condition $f'(0) = -\lambda$ and the initial condition $f''(0) = \alpha$. The implicit relation $h(\alpha, \lambda)$ is defined by the following limit:

$$h(\alpha, \lambda) = \lim_{\eta \to \infty} f'(\eta). \tag{46}$$

The nonuniqueness of solution is shown by defining λ as a function of α when $h(\alpha, \lambda) = 1$,which can be described as an analytic shooting method. Fig. 1 shows the implicit curve of $h(\alpha, \lambda)$. We generate

10 terms to sketch the implicit curve in Fig. 1. It is clear that when $\lambda < \lambda_c = 0.351\ldots$, there are two values of α that satisfy the condition $f'(\infty) = 1$, and only one value of α corresponding to $\lambda = \lambda_c$, and $\lambda = 0$ and no α when $\lambda > \lambda_c$. When $\lambda \to 0$, the left branch of the implicit curve approaches the value of $\alpha = 0.33206$—as it was derived by many authors—while the right branch of the curve approaches the line $\lambda = 0$ asymptotically, which agrees with the fact that when $\lambda = 0$, only one solution exists. The horizontal line represents $\lambda = \lambda_c = 0.354\ldots$.

4. Conclusion

In this article, an analytical solution of the nonhomogeneous Blasius problem is derived using the ADM and the HAM. The solution obtained using ADM converges only for a very restricted domain, $0 \leqslant \eta < \rho_0 \simeq 5.7$. In addition, this solution is half numeric and half analytic. One has to know the value of the initial condition $\alpha = f''(0)$ in order to construct the analytical solution for the problem. On the other hand, the solution obtained using the HAM converges for the whole domain of definition of the function, namely for $0 \leqslant \eta < \infty$.

The question of uniqueness is also addressed. Using the analytical solution obtained by the HAM, we are able to find an implicit relation $h(\alpha, \lambda)$ between the parameters α and λ. The curve of the function $h(\alpha, \lambda) = 1$ indicates that two solutions exist when $0 < \lambda < \lambda_c \simeq 0.354\ldots$, one solution exists for $\lambda = \lambda_c$ and no solution exists for $\lambda > \lambda_c$.

It is also worth mentioning that homotopy analysis has proven to be very efficient in solving these types of problems, especially nonlinear boundary value problems with infinite domain.

References

[1] G. Adomian, A review of the decomposition method in applied mathematics, J. Math. Anal. Appl. 135 (1988) 501–544.

[2] G. Adomian, A review of the decomposition method and some recent results for nonlinear equations, Comput. Math. Appl. 21 (5) (1991) 101–127.

[3] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston, 1994.

[4] G. Adomian, Solution of physical problems by decomposition, Comput. Math. Appl. 27 (9/10) (1994) 145–154.

[5] G. Adomian, Solution of the Thomas–Fermi equation, Appl. Math. Lett. 11 (3) (1998) 131–133.

[6] F.M. Allan, On the similarity solutions of the boundary layer problem over a moving surface, Appl. Math. Lett. 10 (2) (1997) 81–85.

[7] F.M. Allan, R.M. Abu-Saris, On the existence and non-uniqueness of non-homogeneous Blasius problem, Proceedings of the second Pal. International Conference, Gorden and Breach, 1999.

[8] M. Ayub, A. Rasheed, T. Hayat, Exact flow of a third grade fluid past a porous plate using homotopy analysis method, Internat. J. Engrg. Sci. 41 (2003) 2091–2103.

[9] W.A. Coppel, On a differential equation of boundary layer theory, Phil. Trans. Roy. Soc. London, Ser. A 253 (1960) 101–136.

[10] T. Hayat, M. Khan, M. Ayub, On the explicit analytic solutions of an Oldroyd 6-constant fluid, Internat. J. Engrg. Sci. 42 (2004) 123–135.

[11] T. Hayat, M. Khan, S. Asghar, Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid, Acta Mech. 168 (2004) 213–232.

[12] M.Y. Hussaini, W.D. Lakin, Nachman, On similarity solution of a boundary layer problem with upstream moving wall, A., SIAM J. Appl. Math. 7 (4) (1987) 699–709.

[13] S.J. Liao, An explicit totally analytic approximate solution for Blasius viscous flow problems, Internat. J. Non-Linear Mech. 34 (1999) 759–778.
[14] S.J. Liao, A uniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate, J. Fluid Mech. 385 (1999) 101–128.
[15] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman & Hall/CRC Press, Boca Raton, 2003.
[16] S.J. Liao, On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet, J. Fluid Mech. 488 (2003) 189–212.
[17] S.J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147/2 (2004) 499–513.
[18] S.J. Liao, A. Campo, Analytic solutions of the temperature distribution in Blasius viscous flow problems, J. Fluid Mech. 453 (2002) 411–425.
[19] S.J. Liao, K.F. Cheung, Homotopy analysis of nonlinear progressive waves in deep water, J. Engrg. Math. 45 (2) (2003) 105–116.
[20] S.J. Liao, I. Pop, Explicit analytic solution for similarity boundary layer equations, Internat. J. Heat Mass Transfer 47/1 (2004) 75–85.
[21] H. Schlichting, Boundary Layer Theory, seventh ed., McGraw-Hill, New York, 1979.
[22] M. Syam, A domain decomposition method for approximating the solution of the Korteweg–deVries equation, Appl. Math. Comput. (2004), in press.
[23] H. Weyl, On the differential equation of the simplest boundary-layer problems, Ann. Math. 43 (1942) 381–407.