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The application of homotopy analysis method to solve a generalized Hirota–Satsuma coupled KdV equation

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Abstract

Here, an analytic technique, namely the homotopy analysis method (HAM), is applied to solve a generalized Hirota–Satsuma coupled KdV equation. HAM is a strong and easy-to-use analytic tool for nonlinear problems and dose not need small parameters in the equations. Comparison of the results with those of Adomian's decomposition method (ADM) and homotopy perturbation method (HPM), has led us to significant consequences. The homotopy analysis method contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series.

Keywords: Homotopy analysis method; Adomian's decomposition method; Homotopy perturbation method; Generalized Hirota–Satsuma coupled KdV equation

1. Introduction

The investigation of the exact solution to nonlinear equations plays an important role in the study of nonlinear physical phenomena. In this Letter, we consider a generalized Hirota–Satsuma coupled Korteweg–de Vries (KdV) equation which was introduced by Wu et al. [1]. One of the typical equations in the hierarchy is a new generalized Hirota–Satsuma coupled KdV equation as follows:

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x, \\ v_t &= -v_{xxx} + 3uv_x, \\ w_t &= -w_{xxx} + 3uw_x. \end{aligned} \tag{1}$$

Eq. (1) is reduced to a new complex coupled KdV equation [1] and the Hirota–Satsuma equation [2] with $w = v^*$ and $w = v$, respectively. More recently, the soliton solutions for this equation is constructed by Fan [3]. The discussed generalized Hirota–Satsuma coupled KdV equation has been studied by many researcher via different approaches, for example, Jacobi elliptic function method [4], the projective Riccati equations method [5], the Adomian's decomposition method [6] and recently by homotopy perturbation method [7]. For Eq. (1), the obtained results by HPM [7] are as the same results obtained by Adomian's decomposition method [6] and these are valid only for small values of x and t. The absolute error of HPM (and ADM) results for $u(x,t)$ by the 5th-order and 10th-order approximation (for some parameters) where $x \in [0, 100]$ and $t \in [0, T] = [0, 100]$ are plotted in Figs. 1 and 2, respectively. The same situations exist for $v(x,t)$ and $w(x,t)$. This shows, the limitations of the HPM and ADM for the considered problem.

In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely *homotopy analysis method* (HAM), [8–17]. This method has been successfully applied to solve many types of nonlinear problems by others [18–25].

In this Letter, the basic idea of the HAM is introduced and then its application in a generalized Hirota–Satsuma coupled KdV equation is studied. Also, the comparison is made with the exact solution and ADM [26] and HPM, which is obtained by [6,7]. The homotopy analysis method contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and

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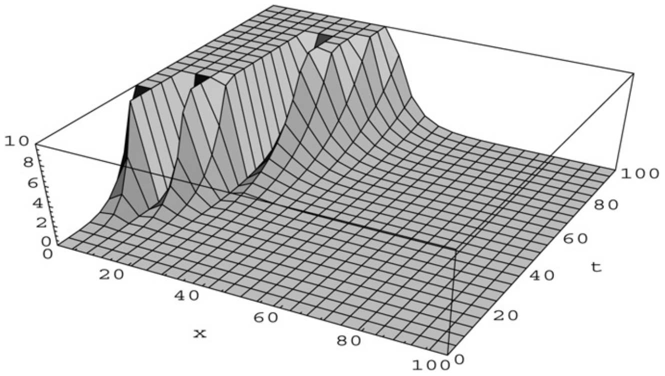

Fig. 1. Absolute error for the 5th-order approximation by HPM and ADM.

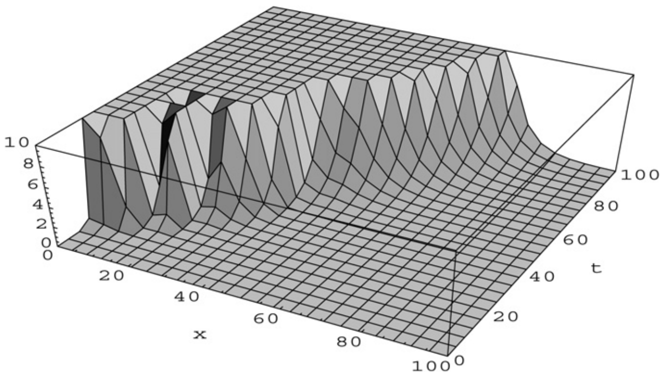

Fig. 2. Absolute error for the 10th-order approximation by HPM and ADM.

control the convergence region of solution series for any values of x and t.

2. Basic idea of HAM

In this Letter, we apply the homotopy analysis method [8–13] to the discussed problem. To show the basic idea, let us consider the following differential equation

$$\mathcal{N}\big[z(x,t)\big]=0,$$

where $\mathcal{N}$ is a nonlinear operator, x and t denote independent variables, $z(x,t)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [10] constructs the so-called zero-order deformation equation

$$(1-p)\mathcal{L}\big[\phi(x,t;p)-z_0(x,t)\big]=p\hbar\mathcal{N}\big[\phi(x,t;p)\big], \tag{2}$$

where $p\in[0,1]$ is the embedding parameter, $\hbar\neq 0$ is a nonzero auxiliary parameter, $\mathcal{L}$ is an auxiliary linear operator, $z_0(x,t)$ is an initial guess of $z(x,t)$, $\phi(x,t;p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p=0$ and $p=1$, it holds

$$\phi(x,t;0)=z_0(x,t), \qquad \phi(x,t;1)=z(x,t),$$

respectively. Thus as p increases from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial guess $z_0(x,t)$ to the solution $z(x,t)$. Expanding $\phi(x,t;p)$ in Taylor series with respect to p, one has

$$\phi(x,t;p)=z_0(x,t)+\sum_{m=1}^{+\infty}z_m(x,t)p^m, \tag{3}$$

where

$$z_m(x,t)=\frac{1}{m!}\frac{\partial^m\phi(x,t;p)}{\partial p^m}\bigg|_{p=0}. \tag{4}$$

If the auxiliary linear operator, the initial guess, and the auxiliary parameter $\hbar$ are so properly chosen, the series (3) converges at $p=1$, one has

$$z(x,t)=z_0(x,t)+\sum_{m=1}^{+\infty}z_m(x,t),$$

which must be one of solutions of original nonlinear equation, as proved by Liao [10]. As $\hbar=-1$, Eq. (2) becomes

$$(1-p)\mathcal{L}\big[\phi(x,t;p)-z_0(x,t)\big]+p\mathcal{N}\big[\phi(x,t;p)\big]=0,$$

which is used mostly in the homotopy perturbation method, whereas the solution obtained directly, *without using Taylor series* [27,28]. The comparison between HAM and HPM, can be found in [14,29].

According to (4), the governing equation can be deduced from the zero-order deformation equation (2). Define the vector

$$\vec{z}_n=\big\{z_0(x,t),z_1(x,t),\ldots,z_n(x,t)\big\}.$$

Differentiating Eq. (2) m times with respect to the embedding parameter p and then setting $p=0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}\big[z_m(x,t)-\chi_m z_{m-1}(x,t)\big]=\hbar R_m(\vec{z}_{m-1}), \tag{5}$$

where

$$R_m(\vec{z}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(x,t;p)]}{\partial p^{m-1}}\bigg|_{p=0}, \tag{6}$$

and

$$\chi_m=\begin{cases}0, & m\leqslant 1,\\ 1, & m>1.\end{cases}$$

It should be emphasized that $z_m(x,t)$ for $m\geqslant 1$ is governed by the linear equation (5) with the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

3. Applications

To investigate the traveling wave solution of Eq. (1) and to made comparison with HPM [7] and ADM [6], we choose the linear operator

$$\mathcal{L}\big[\phi(x,t;p)\big]=\frac{\partial\phi(x,t;p)}{\partial t},$$

with the property

$$\mathcal{L}[c]=0,$$

where c is constant. From (1), we define a system of nonlinear operators as

$$\begin{aligned}
&\mathcal{N}_1\big[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)\big]\\
&\quad=\frac{\partial\phi_1(x,t;p)}{\partial t}-\frac{1}{2}\frac{\partial^3\phi_1(x,t;p)}{\partial x^3}\\
&\qquad+3\phi_1(x,t;p)\frac{\partial\phi_1(x,t;p)}{\partial x}\\
&\qquad-3\frac{\partial}{\partial x}\big(\phi_2(x,t;p)\phi_3(x,t;p)\big),\\
&\mathcal{N}_2\big[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)\big]\\
&\quad=\frac{\partial\phi_2(x,t;p)}{\partial t}+\frac{\partial^3\phi_2(x,t;p)}{\partial x^3}-3\phi_1(x,t;p)\frac{\partial\phi_2(x,t;p)}{\partial x},\\
&\mathcal{N}_3\big[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)\big]\\
&\quad=\frac{\partial\phi_3(x,t;p)}{\partial t}+\frac{\partial^3\phi_3(x,t;p)}{\partial x^3}-3\phi_1(x,t;p)\frac{\partial\phi_3(x,t;p)}{\partial x}.
\end{aligned}\tag{7}$$

Using above definition, we construct the *zeroth- order deformation equations*

$$\begin{aligned}
&(1-p)\mathcal{L}\big[\phi_i(x,t;p)-z_{i,0}(x,t)\big]\\
&\quad=p\hbar_i\mathcal{N}_i\big[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)\big],\quad i=1,2,3.
\end{aligned}$$

Obviously, when $p=0$ and $p=1$,

$$\begin{aligned}
&\phi_1(x,t;0)=z_{1,0}(x,t)=u(x,0), && \phi_1(x,t;1)=u(x,t),\\
&\phi_2(x,t;0)=z_{2,0}(x,t)=v(x,0), && \phi_2(x,t;1)=v(x,t),\\
&\phi_3(x,t;0)=z_{3,0}(x,t)=w(x,0), && \phi_3(x,t;1)=w(x,t).
\end{aligned}$$

Therefore, as the embedding parameter p increases from 0 to 1, $\phi_i(x,t;p)$ varies from the initial guess $z_{i,0}(x,t)$ to the solution $u(x,t)$, $v(x,t)$ and $w(x,t)$, for $i=1,2,3$, respectively. Expanding $\phi_i(x,t;p)$ in Taylor series with respect to p for $i=1,2,3$, one has

$$\phi_i(x,t;p)=z_{i,0}(x,t)+\sum_{m=1}^{+\infty}z_{i,m}(x,t)p^m,$$

where

$$z_{i,m}(x,t)=\frac{1}{m!}\frac{\partial^m\phi_i(x,t;p)}{\partial p^m}\bigg|_{p=0}.$$

If the auxiliary linear operator, the initial guess, and the auxiliary parameters $\hbar_i$ are so properly chosen, the above series converge at $p=1$, one has

$$\begin{aligned}
u(x,t)&=z_{1,0}(x,t)+\sum_{m=1}^{+\infty}z_{1,m}(x,t),\\
v(x,t)&=z_{2,0}(x,t)+\sum_{m=1}^{+\infty}z_{2,m}(x,t),\\
w(x,t)&=z_{3,0}(x,t)+\sum_{m=1}^{+\infty}z_{3,m}(x,t),
\end{aligned}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [10]. Define the vectors

$$\vec{z}_{i,n}=\big\{z_{i,0}(x,t),z_{i,1}(x,t),\ldots,z_{i,n}(x,t)\big\},\quad i=1,2,3.$$

We gain the *mth-order deformation equations*

$$\begin{aligned}
&\mathcal{L}\big[z_{i,m}(x,t)-\chi_m z_{i,m-1}(x,t)\big]\\
&\quad=\hbar_i R_{i,m}(\vec{z}_{1,m-1},\vec{z}_{2,m-1},\vec{z}_{3,m-1}),\quad i=1,2,3,
\end{aligned}\tag{8}$$

where

$$\begin{aligned}
&R_{1,m}(\vec{z}_{1,m-1},\vec{z}_{2,m-1},\vec{z}_{3,m-1})\\
&\quad=\frac{\partial z_{1,m-1}(x,t)}{\partial t}-\frac{1}{2}\frac{\partial^3 z_{1,m-1}(x,t)}{\partial x^3}\\
&\qquad+3\sum_{n=0}^{m-1}z_{1,n}(x,t)\frac{\partial z_{1,m-1-n}(x,t)}{\partial x}\\
&\qquad-3\frac{\partial}{\partial x}\left(\sum_{n=0}^{m-1}z_{2,n}(x,t)z_{3,m-1-n}(x,t)\right),\\
&R_{2,m}(\vec{z}_{1,m-1},\vec{z}_{2,m-1},\vec{z}_{3,m-1})\\
&\quad=\frac{\partial z_{2,m-1}(x,t)}{\partial t}+\frac{\partial^3 z_{2,m-1}(x,t)}{\partial x^3}\\
&\qquad-3\sum_{n=0}^{m-1}z_{1,n}(x,t)\frac{\partial z_{2,m-1-n}(x,t)}{\partial x},\\
&R_{3,m}(\vec{z}_{1,m-1},\vec{z}_{2,m-1},\vec{z}_{3,m-1})\\
&\quad=\frac{\partial z_{3,m-1}(x,t)}{\partial t}+\frac{\partial^3 z_{3,m-1}(x,t)}{\partial x^3}\\
&\qquad-3\sum_{n=0}^{m-1}z_{1,n}(x,t)\frac{\partial z_{3,m-1-n}(x,t)}{\partial x}.
\end{aligned}$$

Now, the solution of the mth-order deformation Eqs. (8) for $m\geqslant 1$ become

$$\begin{aligned}
z_{i,m}(x,t)&=\chi_m z_{i,m-1}(x,t)\\
&\quad+\hbar_i\mathcal{L}^{-1}\big[R_{i,m}(\vec{z}_{1,m-1},\vec{z}_{2,m-1},\vec{z}_{3,m-1})\big],
\end{aligned}\tag{9}$$

for $i=1,2,3$. For simplicity, we suppose $\hbar_1=\hbar_2=\hbar_3=\hbar$.

Firstly, we consider the solution of Eq. (1) with the initial conditions [3,6,7]

$$\begin{aligned}
u(x,0)&=\frac{1}{3}\big(\beta-2k^2\big)+2k^2\tanh^2(kx),\\
v(x,0)&=-\frac{4k^2c_0(\beta+k^2)}{3c_1^2}+\frac{4k^2(\beta+k^2)}{3c_1}\tanh(kx),\\
w(x,0)&=c_0+c_1\tanh(kx),
\end{aligned}\tag{10}$$

where k, c_0, $c_1\neq 0$, and β are arbitrary constants. According to (9) and (10), we now successively obtain

$$\begin{aligned}
z_{1,1}(x,t)&=-4\hbar\beta k^3t\,\mathrm{sech}^2(kx)\tanh(kx),\\
z_{1,2}(x,t)&=2\hbar^2\beta^2k^4t^2\big(2-\cosh(2kx)\big)\,\mathrm{sech}^4(kx)\\
&\quad-4\big(\hbar+\hbar^2\big)\beta k^3t\,\mathrm{sech}^2(kx)\tanh(kx),\\
z_{2,1}(x,t)&=-\frac{4\hbar\beta k^3(\beta+k^2)t\,\mathrm{sech}^2(kx)}{3c_1},
\end{aligned}$$

$$z_{2,2}(x,t) = -\frac{4\hbar^2\beta^2k^4(\beta+k^2)t^2\,\mathrm{sech}^2(kx)\tanh(kx)}{3c_1} - \frac{4(\hbar+\hbar^2)\beta k^3 t(\beta+k^2)\,\mathrm{sech}^2(kx)}{3c_1},$$
$$z_{3,1}(x,t) = -c_1\hbar\beta kt\,\mathrm{sech}^2(kx),$$
$$z_{3,2}(x,t) = -c_1\hbar^2\beta^2k^2t^2\,\mathrm{sech}^2(kx)\tanh(kx) - c_1(\hbar+\hbar^2)\beta kt\,\mathrm{sech}^2(kx).$$

Obviously, for $\hbar = -1$ the obtained solutions are as the same HPM [7], also ADM [6]. Unfortunately, the HPM and ADM solutions are valid only for some values of x and t, as reported in [6,7].

Using Taylor series with initial conditions (10), we obtain the closed form solutions as follows:

$$u(x,t) = \frac{1}{3}(\beta-2k^2) + 2k^2\tanh^2[k(x+\beta t)],$$
$$v(x,t) = -\frac{4k^2c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)}{3c_1}\tanh[k(x+\beta t)],$$
$$w(x,t) = c_0 + c_1\tanh[k(x+\beta t)],$$

which is bell-type for $u(x,t)$ and kink-type for $v(x,t)$ and $w(x,t)$ and constructed by Fan [3]. The absolute error of HPM (and ADM) results for $u(x,t)$ by the 5th-order and 10th-order approximation when $c_0 = 1.5$, $c_1 = 0.1$, $\beta = 1.5$, $k = 0.1$, $x \in [0,100]$ and $t \in [0,T] = [0,100]$ are plotted in Figs. 1 and 2, respectively.

Also the error of norm 2 with HAM by 10th-order approximation, i.e.,

$$\left(\frac{1}{81}\sum_{i,j}\big(u(x_i,t_j) - z_{1,10}(x_i,t_j)\big)^2\right)^{0.5},$$
$$\left(\frac{1}{81}\sum_{i,j}\big(v(x_i,t_j) - z_{2,10}(x_i,t_j)\big)^2\right)^{0.5},$$
$$\left(\frac{1}{81}\sum_{i,j}\big(w(x_i,t_j) - z_{3,10}(x_i,t_j)\big)^2\right)^{0.5},$$

where $u(x,t)$, $v(x,t)$ and $w(x,t)$ are exact solutions, $x_i = t_i = 5i$, $i = 0,1,\ldots,8$ with respect to $\hbar$ are plotted in Figs. 3–5 for $x \in [0,40]$ and $t \in [0,T] = [0,40]$. When $\hbar$ is a function of T such as $\hbar = -1/(1+T)$, the higher the order of approximation, the smaller absolute error, as shown in Figs. 6–8, indicating that the absolute error about T might become zero as the order of approximation tends to infinity.

To examine the accuracy and reliability of the HAM for the generalized Hirota–Satsuma coupled KdV equation, we can consider the different initial values [3,6,7]

$$u(x,0) = \frac{1}{3}(\beta-8k^2) + 4k^2\tanh^2(kx),$$
$$v(x,0) = -\frac{4k^2(3k^2c_0 - 2\beta c_2 + 4k^2c_2)}{3c_2^2} + \frac{4k^2}{c_2}\tanh^2(kx),$$
$$w(x,0) = c_0 + c_2\tanh^2(kx), \tag{11}$$

where k, c_0, $c_2 \neq 0$, and β are arbitrary constants. Using Taylor series with initial conditions (11), we obtain the closed form

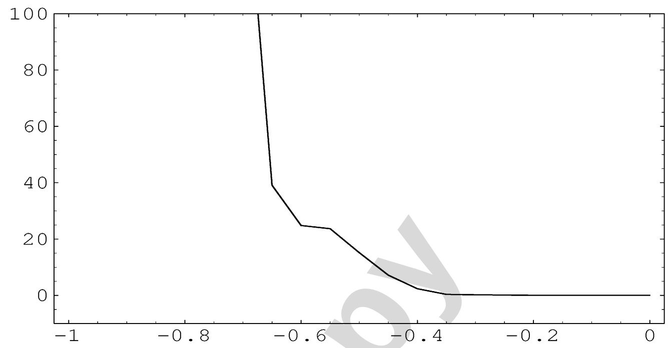

Fig. 3. Error of norm 2 for the 10th-order approximation by HAM for $u(x,t)$ per $\hbar$.

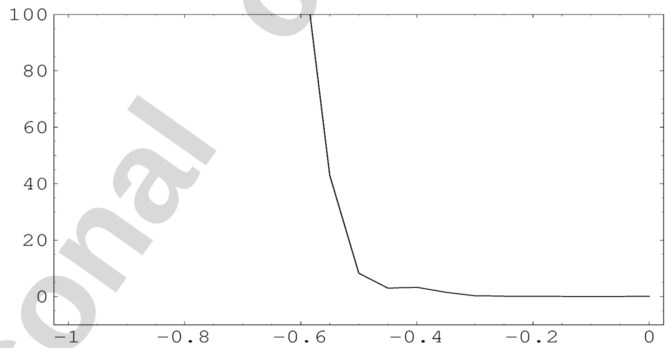

Fig. 4. Error of norm 2 for the 10th-order approximation by HAM for $v(x,t)$ per $\hbar$.

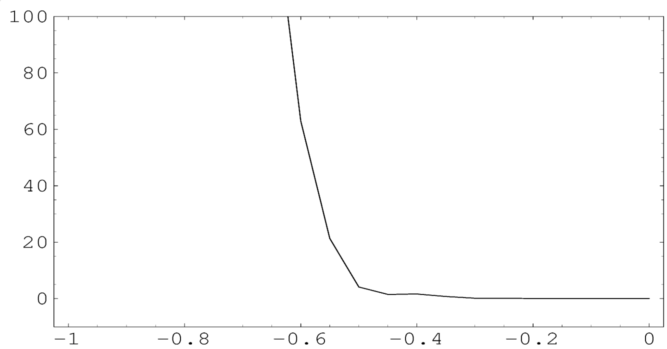

Fig. 5. Error of norm 2 for the 10th-order approximation by HAM for $w(x,t)$ per $\hbar$.

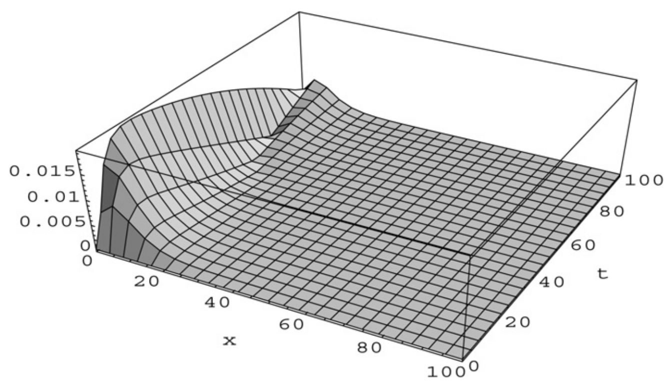

Fig. 6. Absolute error for the 10th-order approximation by HAM for $u(x,t)$ and $\hbar = -1/101$.

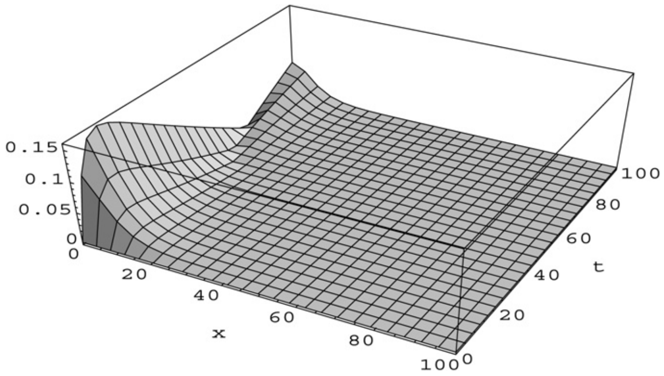

Fig. 7. Absolute error for the 10th-order approximation by HAM for $v(x,t)$ and $\hbar = -1/101$.

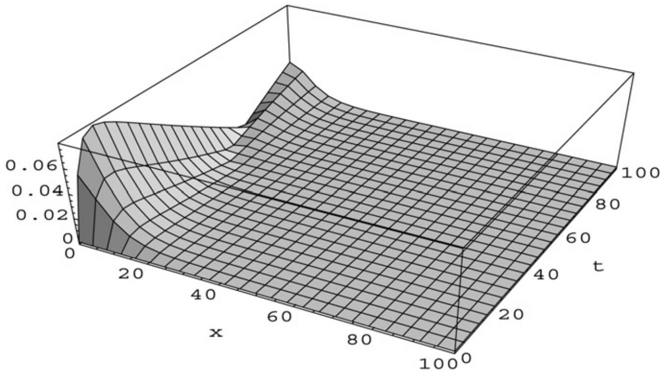

Fig. 8. Absolute error for the 10th-order approximation by HAM for $w(x,t)$ and $\hbar = -1/101$.

solutions as follows:

$$u(x,t) = \frac{1}{3}\left(\beta - 8k^2\right) + 4k^2 \tanh^2\left[k(x+\beta t)\right],$$

$$v(x,t) = -\frac{4k^2(3k^2c_0 - 2\beta c_2 + 4k^2c_2)}{3c_2^2} + \frac{4k^2}{c_2}\tanh^2\left[k(x+\beta t)\right],$$

$$w(x,t) = c_0 + c_2 \tanh^2\left[k(x+\beta t)\right],$$

which is bell-type for all $u(x,t)$, $v(x,t)$ and $w(x,t)$ and constructed by Fan [3]. The same situation exists for this example. For briefly, only the error of norm 2 with HAM by 5th-order approximation with respect to $\hbar$ are plotted in Figs. 9–11 for $x \in [0,40]$ and $t \in [0,T] = [0,40]$, when $c_0 = 1.5$, $c_2 = 0.1$, $\beta = 1.5$, $k = 0.1$.

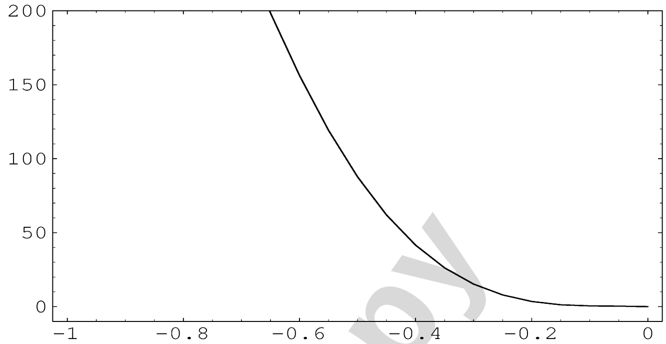

Fig. 9. Error of norm 2 for the 5th-order approximation by HAM for $u(x,t)$ per $\hbar$.

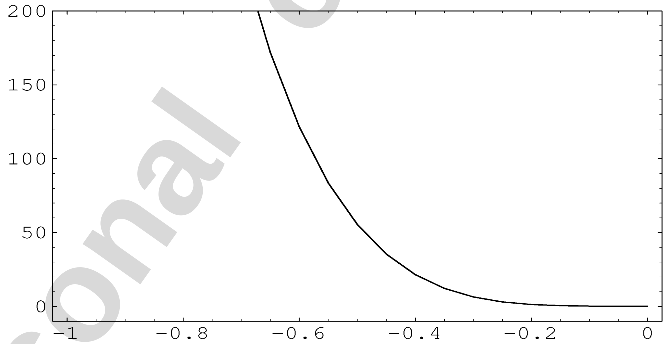

Fig. 10. Error of norm 2 for the 5th-order approximation by HAM for $v(x,t)$ per $\hbar$.

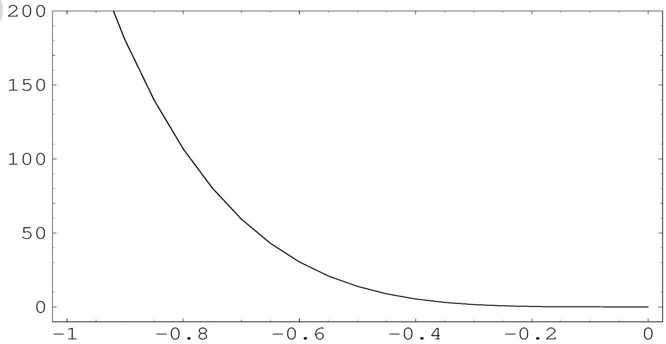

Fig. 11. Error of norm 2 for the 5th-order approximation by HAM for $w(x,t)$ per $\hbar$.

4. Conclusions

In this Letter, the homotopy analysis method (HAM) [10] is applied to obtain the solution of a generalized Hirota–Satsuma coupled KdV equation. HAM provides us with a convenient way to control the convergence of approximation series by adapting $\hbar$, which is a fundamental qualitative difference in analysis between HAM and other methods. Also, it has been shown that the HPM and ADM are valid only for some values of x and t.

Also, according to Figs. 3–5 in first case and Figs. 9–11 in second case, the series solutions obtained by HPM, i.e., HAM with $\hbar = -1$, are divergent which have no meanings [14]. This Letter shows us the validity and great potential of the HAM for nonlinear problems in science and engineering.

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References

[1] Y.T. Wu, X.G. Geng, X.B. Hu, S.M. Zhu, Phys. Lett. A 255 (1999) 259.
[2] R. Hirota, J. Satsuma, Phys. Lett. A 85 (1981) 407.
[3] E. Fan, Phys. Lett. A 282 (2001) 18.
[4] Y. Yu, Q. Wang, H. Zhang, Chaos Solitons Fractals 26 (2005) 1415.
[5] X. Yong, H.Q. Zhang, Chaos Solitons Fractals 26 (2005) 1105.
[6] D. Kaya, Appl. Math. Comput. 147 (2004) 69.
[7] D.D. Ganji, M. Rafei, Phys. Lett. A 356 (2006) 131.
[8] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, PhD thesis, Shanghai Jiao Tong University, 1992.
[9] S.J. Liao, Int. J. Non-Linear Mech. 34 (1999) 759.
[10] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman & Hall/CRC Press, Boca Raton, 2003.
[11] S.J. Liao, J. Fluid Mech. 488 (2003) 189.
[12] S.J. Liao, Appl. Math. Comput. 147 (2004) 499.
[13] S.J. Liao, Int. J. Heat Mass Transfer 48 (2005) 2529.
[14] S.J. Liao, Appl. Math. Comput. 169 (2005) 1186.
[15] S.J. Liao, J. Su, A.T. Chwang, Int. J. Heat Mass Transfer 49 (2006) 2437.
[16] S.J. Liao, E. Magyari, Z. Angew. Math. Phys. 57 (2006) 777.
[17] S.J. Liao, Stud. Appl. Math. 117 (2006) 239.
[18] S. Abbasbandy, Phys. Lett. A 360 (2006) 109.
[19] S. Asghar, M. Mudassar Gulzar, M. Ayub, Acta Mech. Sinica 22 (2006) 195.
[20] M. Ayub, A. Rasheed, T. Hayat, Int. J. Eng. Sci. 41 (2003) 2091.
[21] T. Hayat, M. Khan, Nonlinear Dynamics 42 (2005) 395.
[22] T. Hayat, M. Khan, M. Ayub, Z. Angew. Math. Phys. 56 (2005) 1012.
[23] M. Sajid, T. Hayat, S. Asghar, Phys. Lett. A 355 (2006) 18.
[24] S.P. Zhu, Quant. Finance 6 (2006) 229.
[25] S.P. Zhu, ANZIAM J. 47 (2006) 477.
[26] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic, Dordrecht, 1994.
[27] J.H. He, Int. J. Mod. Phys. B 20 (2006) 1141.
[28] J.H. He, Phys. Lett. A 350 (2006) 87.
[29] J.H. He, Appl. Math. Comput. 156 (2004) 527.