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Comparison of HAM and HPM methods in nonlinear heat conduction and convection equations

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Abstract

Recently, Rajabi et al. (Application of homotopy perturbation method in nonlinear heat conduction and convection equations, Phys. Lett. A 360 (2007) 570–573.) discussed the solutions of temperature distribution in lumped system of combined convection–radiation. They solved a nonlinear equation of the steady conduction in a slab with variable thermal conductivity using both perturbation and homotopy perturbation methods. They claim that homotopy perturbation method (HPM) does not require any small parameter. However, this statement is not true always. Moreover, HPM have no criteria for establishing the convergence of the series solution. In this letter we have explicitly shown that the results of the problem considered in example 2 of (Rajabi, Ganji, Therian, Application of homotopy perturbation method in nonlinear heat conduction and convection equations, Phys. Lett. A 360 (2007) 570–573.) are valid only for $0 \leqslant \varepsilon \leqslant 1$. We have used the homotopy analysis method for finding the more meaningful solution.

Keywords: Homotopy analysis method; Convergence; Heat transfer

1. Introduction

The homotopy analysis method (HAM) has been proposed by Liao in his PhD dissertation in 1992. However, in Liao's PhD dissertation [14], Liao did not introduced the auxiliary parameter $\hbar$, but simply followed the traditional concept of homotopy to construct the following one-parameter family of equations:

$$(1-p)\mathscr{L}(u) + p\mathscr{N}(u) = 0, \tag{1}$$

where $\mathscr{L}$ is an auxiliary linear operator, $\mathscr{N}$ is a nonlinear operator related to the original nonlinear problem $\mathscr{N}(u) = 0$ and p is the embedding parameter. For example, Eq. (1.49) in Liao's dissertation [14] is in the above form. In [15], Liao expressed the above equation in a different form as

$$(1-p)\mathscr{L}(u) = -p\mathscr{N}(u). \tag{2}$$

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For details, please refer to Eq. (2.11) in [15]. However, Liao found that in some cases the series solution is divergent. To overcome this disadvantage, in 1997, Liao introduced the so-called auxiliary parameter in [16,17] to construct the following two-parameter family of equation:

$$(1-p)\mathscr{L}(u-u_0)=\hbar p\mathscr{N}(u), \tag{3}$$

where u_0 is an initial guess. For details, please refer to Eq. (2.1) in [16] and Eq. (2.1) in [17]. Obviously, Eq. (1) is a special case of Eq. (3) when $\hbar=-1$. Liao pointed out [14,17–20,22] that the convergence of the solution series given by the HAM is determined by $\hbar$, and thus one can always get a convergent series solution by means of choosing a proper value of $\hbar$. So, the auxiliary parameter $\hbar$ provides us a simple way to ensure the convergence of HAM series solutions. Using the definition of Taylor series with respect to the embedding parameter p (which is a power series of p), Liao gave a general equations for high-order approximations.

In 1998, Dr. He followed Dr. Liao's early idea to construct the one-parameter family of equation

$$(1-p)\mathscr{L}(u)+p\mathscr{N}(u)=0, \tag{4}$$

which is exactly the same as Liao's early one-parameter family equation (1), and is a special case of Liao's modified two-parameter equation (3) when $\hbar=-1$. From this point of view, there is nothing new in HPM (proposed by Dr. He [11–13]. Different from Liao [14–19], Dr. He directly substituted the power series of p into Eq. (4) to give equations for high-order approximation by equating the coefficients of like power of p. Unfortunately, as proved by Hayat and Sajid in [10] and Sajid et al. in [25,27], these two approaches give the same equations for high-order approximations. This is mainly because the Taylor series of a given function is unique, which is a basic theory in calculus [5]. Thus, nothing is new in Dr. He's approach, except the new name "homotopy perturbation method": Dr. He just employed the early ideas of Liao's HAM. It must be pointed out that some other authors [26] also pointed out this fact. It is very unrealistic that various workers are still claiming that HPM does not require any small parameter. In this letter we are again showing explicitly that the results presented in [24] for example 2 are divergent for the parameter values $\varepsilon>1$ and also for $\varepsilon<1$ one needs more number of iterations to get the convergent solutions and only a solution up to the second term is not reasonable. The arrangement of the paper is as follows.

The HAM [19] solution is presented in Section 2. HAM is a powerful mathematical technique and has been already applied to several nonlinear problems [1–4,6–9,18,20–23,26,28–30]. In Section 3 we have explicitly shown for the considered example that the equations and solutions of HPM can be obtained as a special case of HAM when $\hbar=-1$. Section 4 includes the analysis of results. Section 5 synthesis the concluding remarks.

2. HAM solution

The problem considered in Ref. [24] is:

$$\frac{\mathrm{d}}{\mathrm{d}X}\left[(1+\varepsilon\theta)\frac{\mathrm{d}\theta}{\mathrm{d}X}\right]=0, \tag{5}$$

$$\theta(0)=1, \quad \theta(1)=0. \tag{6}$$

2.1. Zeroth-order deformation problem

The temperature $\theta(X)$ can be expressed by the set of base functions

$$\left\{X^k \middle| k\geqslant 0\right\} \tag{7}$$

in the form of the following series:

$$\theta(X)=a_{0,0}+\sum_{k=0}^{\infty}a_{m,k}X^k, \tag{8}$$

in which $a_{m,k}$ are the coefficients. Invoking the so-called *Rule of solution expressions* for $\theta(X)$ and Eq. (6) the initial guess $\theta_0(X)$ and linear operators $\mathscr{L}$ are

$$\theta_0(X) = 1 - X, \tag{9}$$

$$\mathscr{L}(f) = f'', \tag{10}$$

where

$$\mathscr{L}[C_1 X + C_2] = 0, \tag{11}$$

and C_1, C_2 are the constants and the nonlinear operator is

$$\mathscr{N}\left[\widehat{\theta}(X,p)\right] = \frac{\partial^2 \widehat{\theta}(X,p)}{\partial X^2} + \varepsilon\left[\left(\frac{\partial \widehat{\theta}(X,p)}{\partial X}\right)^2 + \widehat{\theta}(X,p)\frac{\partial^2 \widehat{\theta}(X,p)}{\partial X^2}\right]. \tag{12}$$

The problem at the zeroth-order is

$$(1-p)\mathscr{L}\left[\widehat{\theta}(X,p) - \theta_0(X)\right] = p\hbar\mathscr{N}\left[\widehat{\theta}(X,p)\right], \tag{13}$$

$$\widehat{\theta}(0,p) = 1, \quad \widehat{\theta}(1,p) = 0, \tag{14}$$

where $\hbar$ is the auxiliary nonzero parameter and $p(\in[0,1])$ is an embedding parameter. For $p=0$ and 1, we have

$$\widehat{\theta}(X,0) = \theta_0(X), \quad \widehat{\theta}(X,1) = \theta(X). \tag{15}$$

The initial guess $\theta_0(X)$ tends to $\theta(X)$ as p varies from 0 to 1. Due to Taylor's series expansion

$$\widehat{\theta}(X,p) = \theta_0(X) + \sum_{m=1}^{\infty} \theta_m(X)p^m, \tag{16}$$

where

$$\theta_m(X) = \left.\frac{1}{m!}\frac{\partial^m \widehat{\theta}(X,p)}{\partial p^m}\right|_{p=0}, \tag{17}$$

and the convergence of series (16) depends upon the values of $\hbar$. The value of $\hbar$ is chosen in such a way that series (16) is convergent at $p=1$. Then by using Eq. (15) one obtains

$$\theta(X) = \theta_0(X) + \sum_{m=1}^{\infty} \theta_m(X). \tag{18}$$

2.2. *mth-order deformation problems*

Here we first differentiate Eq. (13) m times with respect to p then divide by $m!$ and setting $p=0$ we obtain

$$\mathscr{L}\left[\theta_m(X) - \chi_m\theta_{m-1}(X)\right] = \hbar\mathscr{R}_m(X), \tag{19}$$

$$\theta_m(0) = \theta_m(1) = 0, \tag{20}$$

where

$$\mathscr{R}_m(X) = \frac{\mathrm{d}^2\theta_{m-1}}{\mathrm{d}X^2} + \varepsilon\sum_{k=0}^{m-1}(\theta'_{m-1-k}\theta'_k + \theta_{m-1-k}\theta''_k), \tag{21}$$

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{22}$$

The general solutions of Eqs. (19)–(21) can be written as

$$\theta_m(X) = \theta_m^{\star}(X) + C_1 X + C_2, \tag{23}$$

where $\theta_m^{\star}(X)$ is the particular solution and the constants are determined by the boundary conditions (20). In the next section, the linear non-homogeneous Eqs. (19)–(21) are solved using Mathematica in the order $m = 1, 2, 3, \ldots$.

3. Comparison of HPM and HAM

To make a comparison between HPM and HAM we will first show that the first and second order equations of the HAM are exactly the same as presented in Ref. [24] for $\hbar = -1$. Assuming $m = 1$ and $\hbar = -1$, Eqs. (19)–(21) read as

$$\frac{d^2\theta_1}{dX^2} + \varepsilon\left(\frac{d\theta_0}{dX}\frac{d\theta_0}{dX} + \theta_0\frac{d^2\theta_0}{dX^2}\right) = 0, \tag{24}$$

$$\theta_1(0) = \theta_1(1) = 0, \tag{25}$$

which are exactly same as Eqs. (42) and (43) of Ref. [24] when we replace θ by v. It shows that the first order equation of HPM is exactly the same as the first-order equation of HAM when $\hbar = -1$. For $m = 2$ and $\hbar = -1$ we have from Eqs. (19)–(21) that

$$\frac{d^2\theta_2}{dX^2} + \varepsilon\sum_{k=0}^{1}\left(2\frac{d\theta_1}{dX}\frac{d\theta_0}{dX} + \theta_1\frac{d^2\theta_0}{dX^2} + \theta_0\frac{d^2\theta_1}{dX^2}\right) = 0, \tag{26}$$

$$\theta_2(0) = \theta_2(1) = 0, \tag{27}$$

which are again exactly the same as Eqs. (44) and (45) of Ref. [24]. We have checked explicitly for first two orders of approximations that HPM equations are same as HAM equations when $\hbar = -1$. It can be easily checked for higher orders, i.e. for $m = 3, 4, 5, \ldots$. Moreover, the solutions provided by HAM for the first two orders are exactly the same as presented in Eq. (46) of Ref. [24] when $\hbar = -1$. Hence in general we can reduce the HPM equations as well as solutions from the HAM equations and solutions by taking $\hbar = -1$.

4. Analysis of the results

We note that the explicit, analytic expression in Eq. (14) is the series solution of the problem. As pointed out earlier that the convergence region and rate of approximation strongly depend on the choice of the values of the auxiliary parameter $\hbar$ for the HAM. For this purpose, the $\hbar$-curves are plotted for four different values of the parameter ε in Fig. 1. This figure depicts that the interval for admissible values of $\hbar$ shrinks towards zero by increasing ε. As it is proved explicitly for this example in the previous section that the results of HPM can be obtained as a special case of HAM when $\hbar = -1$. Therefore, the results presented in figures and table when $\hbar = -1$ can be considered as results of HPM. However, Fig. 1 shows that $\hbar = -1$ is valid only for $\varepsilon \leqslant 1$. Therefore, one cannot get convergent results using HPM for values $\varepsilon > 1$. To confirm this the ε–$\hbar$ curves for different values of $\hbar$ are drawn against ε in Fig. 2. Fig. 2 elucidates that for large values of ε one has to take small values of the parameter $\hbar$ and also $\hbar = -1$ is not valid for all the ranges of ε. Further to see the convergence of the solutions of HAM we have made Table 1. Table 1 shows that even for $\varepsilon = 0.5$ one can get convergent results for HPM but one needs 20-term solution for that and only a two-term solution is not enough. Also for large values of ε the HPM results are divergent and are shown in Table 1.

5. Concluding remarks

In this letter it has been explicitly proved that for the nonlinear problems where $\hbar$ is different from –1 one cannot get convergent results by using HPM. However, HAM provides us a simple way to control and adjust the convergence regions where and whenever necessary. It is further shown by the solution of example 2 in [24] that HPM is only valid for the week nonlinearity like the traditional perturbation technique. It is also pointed out that claim of HPM solutions as convergent ones for all nonlinear problems is erroneous.

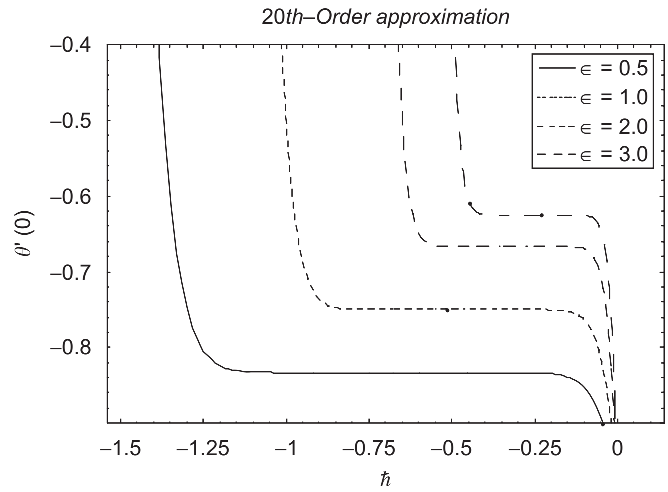

Fig. 1. $\hbar$-curves for different values of ε.

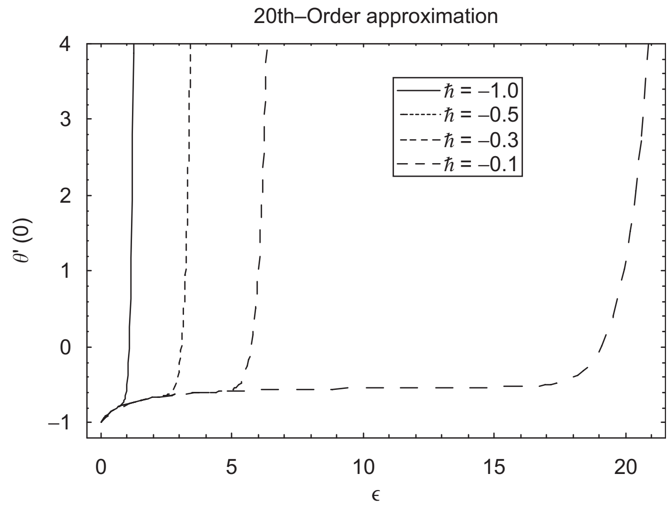

Fig. 2. ε–$\hbar$-curves for different values of $\hbar$.

Table 1
Values of $\theta'(0)$ for different orders of approximation

Order of approximation	$\varepsilon = 0.5$	$\varepsilon = 2.0$		$\varepsilon = 5.0$	
	$\hbar = -1$	$\hbar = -0.25$	$\hbar = -1$	$\hbar = -0.1$	$\hbar = -1$
1	−0.75	−0.75	0.00	−0.75	1.5
5	−0.828125	−0.666992	10.0	−0.5876	1301.5
10	−0.833496	−0.666668	−342	−0.583377	-4.07×10^6
15	−0.833328	−0.666667	1.09×10^4	−0.583334	1.27×10^{10}
20	−0.833333	−0.666667	-3.40×10^5	−0.583333	-3.97×10^{13}
25	−0.833333	−0.666667	1.12×10^7	−0.583333	1.24×10^{17}
30	−0.833333	−0.666667	3.58×10^8	−0.583333	-3.88×10^{20}

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