Homotopy solution for nonlinear differential equations in wave propagation problems

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Abstract

In this paper, we derive a general series solution for nonlinear differential equations based on an alternate form of the homotopy analysis method. The conventional approach begins with a zero-order deformation equation, which includes an auxiliary operator for mapping of an initial approximation to the exact solution and an auxiliary parameter to ensure convergence of the series solution. We express the general series solution directly from the zero-order deformation equation in terms of the Bell polynomial and introduce a new dimension to the convergence characteristics through a second auxiliary parameter. Convergence theorems are provided to assure mapping to the correct solution in the new homotopy defined by two auxiliary parameters. Implementation of the general solution is demonstrated with the periodic long-wave problem governed by the Korteweg de Vries equation and the propagation of high-frequency waves in a relaxing medium given by the Vakhnenko equation. Comparison of the present and exact solutions confirms the effectiveness and validity of the proposed approach. The use of two auxiliary parameters substantially improves the convergence region and rate and provides series solutions to highly nonlinear equations with fewer terms.

1. Introduction

The theory of homotopy, which has found application in differential geometry, can be traced back to the work of Poincaré in 1900 [1]. With the advent of modern computers, the theory re-emerged as the foundation for a class of numerical techniques for the solution of nonlinear equations [2–5]. The basic idea is to map an initial approximation to the exact solution through a homotopy function involving an auxiliary operator and an embedding parameter. A series of problems are generated by gradually varying the embedding parameter and solved recursively using an iterative technique for a numerical solution.

The advance in symbolic computation has enabled a parallel development to provide analytical solutions for nonlinear equations. Liao [6,7] employed the fundamental concept of homotopy to provide a continuous mapping of an initial approximation to the exact solution through a series of deformation equations. This analytic technique, which is known as the homotopy analysis method, has been applied to nonlinear equations in viscous flow [8,9], free surface flow [10,11], wave propagation [12–16], non-Newtonian fluids [17], and quantitative finance [18,19]. Instead of using a curve-tracking algorithm as in [2–5], the homotopy analysis method introduces an auxiliary parameter in the homotopy function to ensure convergence of the solution series. The auxiliary parameter is a key feature of the homotopy analysis method that provides the freedom to choose initial approximations and auxiliary operators for a converging solution [20–22].
Implementation of the homotopy analysis method typically begins with the zero-order deformation equation and requires a detailed derivation to arrive at the series solution for a specific nonlinear problem [6–22]. The method may require higher-order approximations for strongly nonlinear equations resulting in lower accuracy or increasing order of the series solution. Any improvement to the convergence characteristics and implementation of the homotopy analysis method will be of interest to scientists and engineers in applied and computational mathematics fields. In this paper, we enhance the convergence characteristics by introducing a second auxiliary parameter and facilitate the implementation by deriving a general series solution from the zero-order deformation equation. Convergence theorems are provided to assure broad application of the general solution to nonlinear differential equations.

We demonstrate the implementation of the general series solution in wave propagation problems involving the Korteweg de Vries (KdV) equation and the Vakhnenko equation. Their nonlinear characteristics and exact solutions allow examination and verification of the proposed method. The KdV equation is a classical mathematical model for nonlinear long waves. Its cnoidal wave solution is given by the Jacobi elliptic function involving an implicit dispersion relation, which needs to be solved iteratively. An alternative formulation with an explicit solution would therefore be of interest to researchers. The Vakhnenko equation is intriguing because it has a loop soliton solution, which is a highly nonlinear, multi-valued function. The conventional homotopy analysis method has provided analytical solutions for the Vakhnenko equation, but shows some limited convergence characteristics [13]. This problem is revisited here to demonstrate the advantages of the proposed method with strongly nonlinear problems.

2. Homotopy analysis with two auxiliary parameters

The zero-order deformation equation with a single auxiliary parameter has remained the basis of the homotopy analysis method since its inception [6–22]. In this section, we introduce a new form of the zero-order deformation equation with two auxiliary parameters and derive a general series solution for nonlinear differential equations based on the homotopy analysis method. For generality, a nonlinear differential equation is written as

\[ N[u(x,t)] = 0, \]  

where \( N \) is a nonlinear operator, \( u(x,t) \) is an unknown function, and \( x \) and \( t \) denote spatial and temporal independent variables, respectively.

In homotopy, the solution of Eq. (1) is mapped to a function \( \Phi(x,t;q) \) such that, as the embedding parameter \( q \) increases from 0 to 1, \( \Phi(x,t;q) \) varies continuously from an initial approximation \( u_0(x,t) \) to the exact solution \( [2–5] \). Following the approach of Liao [6], we propose the following zero-order deformation equation for the mapping:

\[ [1 - \sigma q + (\sigma - 1)q^2] [L[\Phi(x,t;q) - u_0(x,t)]] = qhN[\Phi(x,t;q)], \]  

where the linear operator \( L[0] = 0 \) facilitates the variation of the initial approximation to the exact solution and the auxiliary parameters \( h \neq 0 \) and \( \sigma \) control the convergence rate and region. When \( q = 0 \), the zero-order deformation Eq. (2) becomes the starting system:

\[ L[\Phi(x,t;0) - u_0(x,t)] = 0, \]  

which gives the initial approximation as the solution

\[ \Phi(x,t;0) = u_0(x,t). \]  

When \( q = 1 \), since \( h = 0 \), the zero-order deformation equation becomes

\[ N[u(x,t;1)] = 0, \]  

which is the target system identical to Eq. (1). With the embedding parameter \( q \) varying from 0 to 1, the zero-order deformation equation changes from the starting system (3) to the target system (5) providing a homotopy between the initial approximation and the exact solution.

In the homotopy analysis method, the nonzero auxiliary parameter \( h \) on the right-hand side of Eq. (2) ensures convergence of the series solution [20–22]. Apart from the new auxiliary parameter \( \sigma \), the left-hand side of Eq. (2) is a quadratic function of the embedding parameter \( q \). The second auxiliary parameter \( \sigma \) adds a new dimension to the homotopy between the starting and target systems and enhances both the convergence rate and region of the solution. When the auxiliary parameter \( \sigma = 1 \), both sides of Eq. (2) are linear functions of the embedding parameter \( q \) and Eq. (2) reduces to the zero-order deformation equation proposed by Liao [6,7]:

\[ [1 - q] [L[\Phi(x,t;q) - u_0(x,t)]] = qhN[\Phi(x,t;q)], \]  

and used by many researchers [6–22]. For the special case of \( h = -1 \) and \( \sigma = 1 \), Eq. (2) reduces to

\[ [1 - q] [L[\Phi(x,t;q) - u_0(x,t)]] + qN[\Phi(x,t;q)] = 0, \]  

which is the traditional homotopy function in [2–5]. Both Liao [6,7] and the traditional homotopy function are special cases of the proposed zero-order deformation Eq. (2), which represents a long line of development that can be traced back to the work of Poincaré [1].
The zero-order deformation Eq. (2) defines a homotopy between the initial approximation \( u_0(x, t) \) and the exact solution \( u(x, t) \) via the auxiliary parameters \( h \) and \( \tau \). The mapping to the exact solution is implemented through a successive approximation with the initial approximation as the first term. To this end, the mapping function \( \Phi(x, t; q) \) is expanded in a Taylor series about \( q = 0 \) as

\[
\Phi(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)q^m,
\]

where

\[
u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \Phi(x, t; q)}{\partial q^m} \right|_{q=0}.
\]

If the initial approximation \( u_0(x, t) \) and the linear operator \( \mathcal{L} \) are properly chosen, the series (8) converges at \( q = 1 \) to give the exact solution

\[
u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t).
\]

This expression provides a relationship between the initial approximation \( u_0(x, t) \) and the exact solution \( u(x, t) \) by means of the higher-order terms \( u_m(x, t) \) defined by Eq. (9).

The term \( u_m(x, t) \) may be derived from the \( m \)th-order deformation equation, which is obtained by differentiating the zero-order deformation Eq. (2) \( m \) times with respect to the embedding parameter \( q \) as:

\[
\frac{d^m}{dq^m} \left[ \mathcal{L} \Phi(x, t; q) \right] = \tau \frac{d^m}{dq^m} \left[ \Phi(x, t; q) \right] + \left( \tau - 1 \right) \frac{d^m}{dq^m} \left[ q^2 \mathcal{L} \Phi(x, t; q) \right] + \tau \mathcal{L} u_0(x, t) \frac{d^m}{dq^m} q - \left( \tau - 1 \right) \mathcal{L} u_0(x, t) \frac{d^m}{dq^m} q^2 = \mathcal{L} \frac{d^m}{dq^m} \left[ q^N \Phi(x, t; q) \right] - \mathcal{L} \frac{d^m}{dq^m} \left[ q^N u_0(x, t) \right].
\]

Setting the embedding parameter \( q = 0 \) and then dividing the resulting expression by \( m! \) gives a series of linear differential equations in terms of \( u_m(x, t) \) as

\[
\mathcal{L} \left[ u_m(x, t) - \chi_m \tau u_{m-1}(x, t) + Z_{m-1} (\tau - 1) u_{m-2}(x, t) \right] = \mathcal{L} R_m(x, t),
\]

in which

\[
R_m(x, t) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\Phi(x, t; q)]}{\partial q^{m-1}} \right|_{q=0}.
\]

where \( \chi_m = 0 \) for \( m \leq 1 \) and \( \chi_m = 1 \) for \( m > 1 \). Expansion of the derivative of the composite function \( N[\Phi(x, t; q)] \) in Eq. (13) by Faà di Bruno’s formula yields

\[
R_m(x, t) = \frac{1}{(m-1)!} \sum_{k=1}^{m-1} \frac{\partial^k}{\partial q^k} \left[ u_0(x, t) \right] B_{m-1,k} \left[ u_1, 2u_2, \ldots, (m-k)u_{m-k} \right],
\]

where the superscript \( k \) in parentheses indicates the order of derivative with respect to \( u_0 \) and \( B_{m-1,k} \) denotes the Bell polynomials given by

\[
B_{m-1,k}[u_1, 2u_2, \ldots, (m-k)u_{m-k}] = \sum_{j_1+j_2+\cdots+j_m=k} \frac{(m-1)!}{j_1!j_2!\cdots j_{m-k}!} (u_1)^{j_1}(u_2)^{j_2}\cdots(u_{m-k})^{j_m}.
\]

The summation in the Bell polynomials is over all nonnegative integers \( j_1, j_2, \ldots, j_{m-k} \) for which \( j_1 + j_2 + \cdots + j_{m-k} = k \) and \( j_1 + 2j_2 + \cdots + (m-k)j_{m-k} = m - 1 \).

We have converted the nonlinear differential equation (1) into a series of linear equation (12). Inverting the linear operator gives the general series solution for the nonlinear equation as

\[
u_m(x, t) = \tilde{u}(x, t) + \chi_m \tau \nu_{m-1}(x, t) - Z_{m-1} (\tau - 1) \nu_{m-2}(x, t) + \mathcal{L}^{-1} \left[ R_m(x, t) \right],
\]

where \( \tilde{u}(x, t) \) is the general solution of the linear differential equation \( \mathcal{L} u(x, t) = 0 \). Note that Eq. (16) does not include the embedding parameter \( q \) anymore and can be calculated recursively through \( u_0(x, t), u_1(x, t), \ldots, u_{m-1}(x, t) \). In spite of its appearance, the series solution (16) is rather simple in symbolic computation and can be readily implemented using standard functions in Mathematica, Maple, or Matlab for general application.

3. Convergence theorems

The second auxiliary parameter introduces an additional dimension to the mapping of an initial approximation to the solution of a nonlinear equation. The resulting homotopy allows better optimization of the series solution and greater free-
dom in the selection of linear auxiliary operators. A series, however, will be of no use if it does not converge to the solution. It is therefore necessary to prove that, for the proposed zero-order deformation equation, if the series solution (10) is convergent, it converges to the correct solution. In this section, we provide a two-step proof that the general solution (16) leads to the exact solution of the nonlinear equation (1) through convergence of the series solution (10).

**Theorem 1.** As long as the series solution (10) converges,
\[ \sum_{m=1}^{\infty} R_m(x, t) = 0. \]

**Proof.** Summation of Eq. (12) from \( m = 1 \) to \( \infty \) gives
\[ \sum_{m=1}^{\infty} L[u_m(x, t) - \chi_m \sigma u_{m-1}(x, t) + \chi_{m-1}(\sigma - 1)u_{m-2}(x, t)] = \sum_{m=1}^{\infty} hR_m(x, t). \]
Recall that \( \chi_m = 0 \) for \( m \leq 1 \) and \( \chi_m = 1 \) for \( m > 1 \) and \( L \) is a linear operator, the left-hand side can be rearranged to give
\[ L \left( \sum_{m=1}^{\infty} u_m(x, t) - \sigma u_m(x, t) + (\sigma - 1)u_m(x, t) \right) = \sum_{m=1}^{\infty} hR_m(x, t), \]
which becomes
\[ L \left( 0 \cdot \sum_{m=1}^{\infty} u_m(x, t) \right) = \sum_{m=1}^{\infty} hR_m(x, t). \]
Since the series solution (10) is convergent, it is finite and its product with zero is zero. With \( L[0] = 0 \) and \( h \neq 0 \), we have
\[ \sum_{m=1}^{\infty} R_m(x, t) = 0. \]

**Theorem 2.** As long as the series solution (10) converges, it must be a solution of the nonlinear differential Eq. (1).

**Proof.** Let \( \varepsilon(x, t; q) = N[\Phi(x, t; q)] \) denote the residual error of Eq. (1). The residual error at \( q = 1 \) can be expanded by a Taylor series at \( q = 0 \) to give
\[ \varepsilon(x, t; q = 1) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m N[\Phi(x, t; q)]}{dq^m} \bigg|_{q=0} \]
Expansion of the derivative of the composite function \( N[\Phi(x, t, q)] \) by Faà di Bruno’s formula gives the residual error as
\[ \varepsilon(x, t; q = 1) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} N^{(k)}[u_0(x, t)] R_{m,k}[u_1, 2u_2, \ldots, (m - k + 1)u_{m-k+1}] \]
\[ = \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \sum_{k=0}^{m-1} N^{(k)}[u_0(x, t)] R_{m-1,k}[u_1, 2u_2, \ldots, (m - k)u_{m-k}] \]
\[ = \sum_{m=1}^{\infty} R_m(x, t). \]
Recalling Theorem 1, if the series solution (10) is convergent, we have
\[ \varepsilon(x, t; q = 1) = 0. \]
Therefore, as long as the series solution (10) converges, its residual error is zero and it is a solution of Eq. (1). □

The two convergence theorems assure mapping of the initial approximation to the correct solution in the new homotopy defined by two auxiliary parameters. Theorem 1 also provides an alternative way to test the convergence and accuracy of the series solution (10). Due to Theorem 2, when solving a nonlinear problem, we just need to focus on the selection of the initial approximation and auxiliary linear operator. Once the series converges, it will be a solution of the nonlinear equation.

### 4. Homotopy analysis for wave propagation problems

Many wave propagation problems are defined in the form of a nonlinear differential equation. Once the boundary conditions are defined, a series solution can be readily obtained from Eq. (16). We demonstrate the implementation of the proposed approach with the KdV equation for periodic shallow-water waves and the Vakhnenko equation for high-frequency waves in a relaxing medium. These equations, which include both nonlinearity and dispersion, cover a wide variety of wave
evolution phenomena. Their analytical solutions allow an examination of the two-dimensional convergence characteristics of the general series solution (16).

4.1 The KdV equation

The KdV equation describes the motion of nonlinear gravity waves in shallow water, internal waves in a density-stratified fluid, ion-acoustic waves in plasma, acoustic waves on a crystal lattice, and numerous other phenomena related to wave motion. It is the generic equation for weakly nonlinear long waves and the classical mathematical model for periodic wave or soliton propagation. Many evolution equations, which represent a balance between some form of dispersion and weak nonlinearity, have properties analogous to those of the KdV equation. In physical variables and stationary coordinates, the KdV equation takes the form

\[ \frac{\partial \eta}{\partial t} + C_0 \left( 1 + \frac{3\eta}{2d} \right) \frac{\partial \eta}{\partial x} + \frac{C_0 d^2}{6} \frac{\partial^3 \eta}{\partial x^3} = 0, \]  

where \( g \) is gravitational acceleration, \( d \) water depth, \( \eta \) wave elevation, and \( C_0 = \sqrt{gd} \) is celerity [24,25]. As a classical equation, the mathematical theory behind the KdV equation is rich and is considered as a canonical example to test new methods. A notable example is the inverse scattering transform method, which was developed initially from solving the KdV equation.

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subject to the boundary condition:

\[ f(0) - f(\pi) = 2, \]  

where \( \lambda = \omega C_0 k, \gamma = k^2 d^2 / 6, \) and \( \mu = 3A/2d \) are the dimensionless celerity, wavelength, and amplitude, respectively. The cnoidal wave solution expresses the wave profile and celerity as

\[ f(\theta) = \xi_2 + (\xi_3 - \xi_2) \text{cn}^2 \left( \frac{K(\kappa)}{\pi} \right), \quad \lambda = 1 + \mu (\xi_1 + \xi_2 + \xi_3), \]  

in which

\[ \xi_1 = -\frac{2E(\kappa)}{\kappa^2 K(\kappa)}, \quad \xi_2 = \frac{2}{K^2} \left( 1 - \frac{E(\kappa)}{K(\kappa)} - \kappa^2 \right), \quad \text{and} \quad \xi_3 = \frac{2}{K^2} \left( 1 - \frac{E(\kappa)}{K(\kappa)} \right), \]  

where \( K(\kappa) \) and \( E(\kappa) \) are the complete elliptic integrals of the first and second kinds and the modulus \( \kappa \) measures the relative importance of nonlinearity and dispersion. The terms \( \xi_2 \) and \( \xi_3 \) correspond to the dimensionless elevations at the crest and trough of the wave profile, respectively. Although the elliptic function and integrals can be calculated with a rapid convergent scheme [29], one still needs to evaluate the modulus \( \kappa \) numerically from

\[ \kappa^2 K^2(\kappa) = \frac{\mu}{6\gamma}, \]  

which is the dispersion relation of the cnoidal wave solution.

We utilize the homotopy analysis method in the form of Eq. (16) to derive an alternative solution to the KdV equation (18) in the form of \( N[f(\theta), \lambda] \). The Mth-order approximation of the wave profile \( f(\theta) \) and celerity \( \lambda \) are expressed, respectively, as

\[ f(\theta) = f_0(\theta) + \sum_{m=1}^{M} f_m(\theta), \]  

\[ \lambda = \sum_{m=1}^{M} \lambda_m. \]  

Considering the boundary condition (19) and the periodic solution we are seeking, we choose \( f_0(\theta) = \cos \theta \) as the initial approximation and expand each higher-order term as a Fourier series

\[ f_m(\theta) = \sum_{k=1}^{N_m(m)} \beta_{m,k} \cos(k\theta), \]  

where \( N_m(m) \) denotes an integer function of \( m \) and \( \beta_{m,k} \) are coefficients to be determined. Although the homotopy analysis method allows great freedom in the selection the auxiliary linear operator, we construct the operator using the linear terms in the KdV equation (18) to retain its characteristics in the mapping:
This linear operator has the general solution for $L[f(\theta)] = 0$ as

$$\tilde{f}(\theta) = C_1 + C_2 \cos \theta + C_3 \sin \theta,$$

where $C_1$, $C_2$ and $C_3$ are constants. The inverse operator is given by:

$$L^{-1}[\sin(\theta)] = -\theta \sin \theta/2$$
$$L^{-1}[\sin(n\theta)] = \cos(n\theta)/n(n^2 - 1) \quad \text{for} \quad n \geq 2$$

where $n$ denotes an integer.

With the linear operator defined and the initial approximation chosen, the general series solution (16) becomes

$$\frac{1}{(m-1)!} \sum_{k=0}^{m-1} N^k(0)B_{m-1,k}[f_1, 2f_2, \ldots, (m-k)f_{m-k}] = f_{m-1} - \sum_{n=1}^{m} \gamma f_{m-n} + \mu \sum_{n=0}^{m-1} f_n f_{m-n-1}$$

$$= \sum_{n=1}^{N_1(m)} \frac{n \pi \sin(n\theta)}{n(n^2 - 1)},$$

where the coefficients $\tilde{\xi}_{m,n}$ are functions of $f_0, f_1, \ldots, f_{m-1}$ and $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$. Substituting Eqs. (25) and (28) into the general solution (16), we have

$$f_m(\theta) = C_1 + C_2 \cos \theta + C_3 \sin \theta + \chi_m \sigma f_{m-1}(\theta) - \chi_{m-1}(\sigma - 1)f_{m-2}(\theta) + \frac{N_1(m)}{n^2 - 1} \sum_{n=1}^{N_1(m)} \frac{n \pi \sin(n\theta)}{n(n^2 - 1)},$$

subject to the boundary condition:

$$f_m(0) - f_m(\pi) = 0.$$  \hspace{1cm} (30)

According to Eq. (23), we have $C_1 = C_2 = 0$. Note that if $R_m(\theta)$ contains the term $\sin \theta$, the solution of the high-order deformation equation (29) will contain $\sin \theta$, which is a secular term of infinity at $\theta = \infty$. Since this does not agree with the boundary condition of a periodic solution, the coefficient $\tilde{\xi}_{m,1}$ of $\sin \theta$ in Eq. (28) must be zero. Eq. (29) becomes

$$f_m(\theta) = \sum_{n=2}^{N_1(m)} \frac{n \pi \sin(n\theta)}{n(n^2 - 1)} + \chi_m \sigma f_{m-1}(\theta) - \chi_{m-1}(\sigma - 1)f_{m-2}(\theta) + C_2 \cos \theta.$$  \hspace{1cm} (31)

The constant $C_2$ is determined by the boundary condition (30) as

$$C_2 = \frac{1}{2} \sum_{n=2}^{N_1(m)} \frac{n \pi \sin(n\pi) - 1}{n(n^2 - 1)}.$$  \hspace{1cm} (32)

Substitute Eq. (23) into Eq. (31), and collect terms of the same order of $\cos(k\theta)$, we have $N_1(m) = m + 1$, and the recursive formulas:

$$\beta_{m,1} = \frac{h}{2} \sum_{n=1}^{m+1} \frac{\tilde{\xi}_{m,m} \cos(n\pi)}{n(n^2 - 1)} + \chi_m \sigma \beta_{m-1,1} - \chi_{m-1}(\sigma - 1)\beta_{m-2,1},$$

$$\beta_{m,k} = \frac{h}{k(k-1)} \sum_{n=1}^{m+1} \frac{\tilde{\xi}_{m,k} \cos(n\pi)}{n(n^2 - 1)} + \chi_m \sigma \beta_{m-1,k} - \chi_{m-1}(\sigma - 1)\beta_{m-2,k} \quad \text{for} \quad k \geq 2.$$  \hspace{1cm} (34)

Here the coefficient $\tilde{\xi}_{m,k}$ is obtained by substituting Eq. (23) into Eq. (28):

$$\tilde{\xi}_{m,k} = \frac{1}{2} h \sum_{j=0}^{m-1} \left\{ \sum_{n=1}^{\min(n+1, m-k)} \beta_{n,k} \beta_{m-n-1,j-k} - \sum_{j=\max(1, k-1)}^{\min(n+1, m-k)} \beta_{n,j} \beta_{m-n-1,j-k} \right\} - (k + \gamma k^3)\beta_{m-1,k} + \sum_{n=1}^{m-1} k \alpha_n \beta_{m-n,k} \quad \text{for} \quad 2 \leq k \leq m - 1$$  \hspace{1cm} (35)

$$\tilde{\xi}_{m,m} = -(m + \gamma m^3)\beta_{m-1,m} + m \lambda_2 \beta_{m-1,m} - \frac{1}{2} \mu \sum_{n=0}^{\min(n+1, m-n)} \sum_{j=\max(1, n)}^{m-n} \beta_{n,j} \beta_{m-n-1,m-j}$$ \hspace{1cm} (36)

$$\tilde{\xi}_{m,m+1} = -\frac{1}{2} \mu \sum_{n=0}^{m-1} (1 + n)\beta_{n,m+1} \beta_{m-n-1,m-n}$$ \hspace{1cm} (37)

With $\tilde{\xi}_{m,1} = 0$, Eq. (28) gives the celerity as
\[ \lambda_m = -\sum_{n=1}^{m-1} \beta_{m-n,1} + \beta_{m-1,1} + \gamma \beta_{m-1,1} + \frac{1}{2} h \sum_{n=0}^{m-1} \sum_{k=0}^{m-n-1} k \beta_{n,k} \beta_{m-n-1,k} \]

\[ -\frac{1}{2} h \sum_{n=0}^{m} \sum_{k=0}^{m-n-1} k \beta_{n,k} \beta_{m-n-1,k} \]

Eqs. (33) and (34) provide the coefficient \( \beta_{m,k} \) in Eq. (23) recursively using only the initial approximation of \( \beta_{0,1} = 1 \). Instead of expressing the solution by the elliptic function and integrals, we provide an alternative formulation to express the wave profile in terms of cosine functions through Eq. (21) and the celerity explicitly through Eq. (22). We can see from Eqs. (21) and (22) that the use of two auxiliary parameters does not increase the number of terms in comparison to the traditional homotopy analysis method, but only modifies the coefficients \( \beta_{m,k} \).

We first examine the solution of a weakly nonlinear problem with \( A/d = 0.005 \) and \( kd = 0.3 \). The wave conditions give rise to \( \kappa^2 = 0.53216 \) with an exact solution of \( \lambda = 0.98515591 \) from the cnoidal wave theory. The auxiliary parameters \( h \) and \( \sigma \) provide a family of solutions from Eqs. (21) and (22). Just as the \( h \)-curves in the traditional homotopy analysis method [6–22], we plot the celerity as a function of \( h \) and \( \sigma \) in Fig. 1 to provide a convenient and reliable way to choose the optimal auxiliary parameters for convergence. The \( h-\sigma \) plot shows a flat region near the center indicating convergence of the solution. The fourth-order approximation has the optimal solution at \( h = -0.59 \) and \( \sigma = 1.0 \), which result in a highly accurate celerity of 0.98515592. A \( \sigma \) of 1.0 indicates that, for weakly nonlinear problems, the present approach reverts to the traditional homotopy analysis method in terms of convergence characteristics. The surface profile in Fig. 2 is almost sinusoidal verifying the validity of the present solution in the truly periodic regime.

The nonlinearity of the problem influences both the solution to the KdV equation and the convergence characteristics of the proposed method. As a demonstration, Fig. 3 shows the computed celerity at the fourth-order of approximation for \( A/d = 0.05 \) and \( kd = 0.3 \). The conditions correspond to \( \kappa^2 = 0.976199 \) and an exact solution of \( \lambda = 0.9982727 \) from the cnoidal wave theory. The \( h-\sigma \) plot shows a similar pattern to the weakly nonlinear case, but with the optimal solution shifted to \( \sigma < 1.0 \). The optimal \( h = -0.50 \) and \( \sigma = 0.8 \) gives a celerity of \( \lambda = 0.998268 \), which is within a 0.0004% error from the exact solution. With \( \sigma = 1.0 \), the conventional homotopy analysis method does not give a satisfactory solution at the fourth-order and requires additional terms for convergence. The corresponding surface elevation in Fig. 4 exhibits more nonlinear characteristics and shows excellent agreement with the cnoidal wave solution. The results also demonstrate the contributions of the second auxiliary parameter \( \sigma \) to the convergence characteristics as nonlinearity increases.

The second auxiliary parameter becomes even more important in the convergence of the series solution for strongly nonlinear problems. The conditions of \( A/d = 0.2 \) and \( kd = 0.3 \) require an eighth-order approximation to produce a satisfactory solution. These nonlinear conditions correspond to \( \kappa^2 = 0.999917 \) and an exact solution of \( \lambda = 1.09541 \) from the cnoidal wave theory. Fig. 5 shows the computed celerity over applicable ranges of auxiliary parameters \( h \) and \( \sigma \). The plot shows a large convergence region with an optimal solution near \( h = -25 \) and \( \sigma = 0.4 \). The calculated celerity equals to 1.09463,
which is within an error of 0.07% of the exact solution. The surface profiles shown in Fig. 6 illustrate the accuracy and efficiency of the series solution in reproducing the narrow wave crest in shallow water with a limited number of terms. As the solution approaches a soliton, the optimal value of $\sigma$ will further decrease to capture the convergence region. This results in the use of fewer terms to provide a converging solution and allows the full potential of the homotopy analysis method to be exploited.

**Fig. 2.** Comparison of wave profiles for $A/d = 0.005$ and $kd = 0.3$. Solid line: present solution at fourth-order; circle: exact solution from the cnoidal wave theory.

**Fig. 3.** The $h-\sigma$ plot for the celerity at fourth-order for $A/d = 0.05$ and $kd = 0.3$. 
4.2 The Vakhnenko equation

The Vakhnenko equation [30] describes the propagation of high-frequency waves in a relaxing medium. Its inverse scattering eigenvalue problem is similar to that of a KdV equation or Boussinesq equation [31,32]. In a stationary coordinate, the Vakhnenko equation takes the form:

\[ \theta/\pi = f(\theta) \]

Fig. 4. Comparison of wave profiles for $A/d = 0.05$ and $kd = 0.3$. Solid line: present solution at fourth-order; circle: exact solution from the cnoidal wave theory.

Fig. 5. The $h-v$ plot for the celerity at eighth-order for $A/d = 0.2$ and $kd = 0.3$. 

4.2 The Vakhnenko equation

The Vakhnenko equation [30] describes the propagation of high-frequency waves in a relaxing medium. Its inverse scattering eigenvalue problem is similar to that of a KdV equation or Boussinesq equation [31,32]. In a stationary coordinate, the Vakhnenko equation takes the form:
\[
\frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) u + u = 0
\]  
(39)

where \( u \) denotes dimensionless pressure. By a coordinate transformation of the Eulerian–Lagrangian type

\[
x = T + \int_{-\infty}^{X} U(Z, T) \, dZ + c_0, \quad t = X,
\]

where \( u(x,t) = U(X,T) \) and \( c_0 \) is a constant, the Vakhnenko equation gives an exact one-loop soliton solution as

\[
\begin{cases}
\frac{u}{\nu} = \frac{3}{2} \sec h^2 \left( \frac{\sqrt{\nu} \zeta}{2} \right), \\
(x - vt)/\sqrt{\nu} = 3 \tan h \left( \frac{\sqrt{\nu} \zeta}{2} \right) - \sqrt{\nu} \zeta,
\end{cases}
\]  
(40)

where \( \nu > 0 \) is celerity and \( \zeta = X - T/\nu \) [33]. This exact solution provides a valuable test for application of the homotopy analysis method in strongly nonlinear problems.

Wu et al. [13] provided a series solution to Eq. (39) by means of the homotopy analysis method with one auxiliary parameter. We re-examine the problem to evaluate the convergence characteristics of the solution with two auxiliary parameters. Through a second transformation from a stationary to a moving coordinate

\[
\int_{-\infty}^{X} U(\zeta, T) \, d\zeta = A + \frac{A}{2} g(\vartheta), \quad \vartheta = \sqrt{\nu} X - T/\sqrt{\nu},
\]

the Vakhnenko equation (39) becomes

\[
\frac{d^3}{d\vartheta^3} g(\vartheta) + r \left( \frac{d}{d\vartheta} g(\vartheta) \right)^2 - \frac{d}{d\vartheta} g(\vartheta) = 0
\]  
(41)

subject to the boundary conditions:

\[
g(0) = -1, \quad g''(0) = 0, \quad g(+\infty) = 0,
\]  
(42)

where \( r \) is a constant to be determined. For the governing Eq. (41) in the form of \( N[g(\vartheta), r] = 0 \), the \( M \)th-order approximation of the solution is expressed as

\[
g(\vartheta) = g_0(\vartheta) + \sum_{m=1}^{M} g_m(\vartheta),
\]  
(43)

\[
r = \sum_{m=1}^{M} r_m.
\]  
(44)
To satisfy the boundary conditions Eq. (42), we select the initial approximation as
\[ g_0(\vartheta) = -\frac{4}{3} \exp(-\vartheta) + \frac{1}{3} \exp(-2\vartheta), \] (45)
and express \( g_m(\vartheta) \) as
\[ g_m(\vartheta) = \sum_{k=1}^{N_2(m)} \alpha_{m,k} \exp(-k\vartheta), \] (46)
where \( N_2(m) \) denotes an integer function of \( m \) and \( \alpha_{m,k} \) are coefficients to be determined. The linear operator, extracted from the nonlinear governing equation (41),
\[ L = \frac{\partial^2}{\partial \vartheta^2} - \frac{\partial}{\partial \vartheta} \] (47)
is applied with the general solution
\[ \tilde{g}(\vartheta) = c_1 \exp(-\vartheta) + c_2 \exp(\vartheta) + c_3, \] (48)
where \( c_1, c_2 \) and \( c_3 \) are constants. The inverse linear operator is given by:
\[ L^{-1}[\exp(-\vartheta)] = \vartheta \exp(-\vartheta)/2 \] (49)
\[ L^{-1}[\exp(-n\vartheta)] = -\exp(-n\vartheta)/n(n^2 - 1) \quad \text{for } n \geq 2 \] (50)
The initial approximation (45) and the linear operator (47) are the same with those in [13] for the purpose of comparison and evaluation.
With the linear operator defined and the initial approximation chosen, the general series solution (16) becomes
\[ g_m(\vartheta) = \chi_m \alpha_{m-1} \exp(-\vartheta) - \chi_{m-1}(\sigma - 1)g_{m-2}(\vartheta) + c_1 \exp(-\vartheta) + c_2 \exp(\vartheta) + c_3 + hL^{-1}[\eta_m(\vartheta)] \] (51)
subject to boundary conditions:
\[ g_m(0) = g''_m(0) = g_m(\infty) = 0. \] (52)
Due to the boundary condition at \( \vartheta = \infty \), we have \( c_2 = c_3 = 0. \) From Eqs. (14) and (50), Eq. (51) becomes:
\[ g_m(\vartheta) = \chi_m \alpha_{m-1} \exp(-\vartheta) - \chi_{m-1}(\sigma - 1)g_{m-2}(\vartheta) + c_1 \exp(-\vartheta) \]
\[ + \sum_{k=1}^{N_2(m-1)} \sum_{n=0}^{N_2(m-2)} \sum_{i=0}^{\min[N_2(n-i),k-1]} \sum_{j=0}^{\max(k-2i-2)} h(-k+j)j_{m-n} \alpha_{n-i, j} \exp(-k\vartheta) \]
\[ \frac{1}{k(k^2 - 1)} + \sum_{k=1}^{N_2(m-1)} \sum_{n=0}^{N_2(n-i),k-1} \sum_{j=0}^{\max(k-2i-2)} h\alpha_{m-1,k} \exp(-k\vartheta) \] (53)
Substituting Eq. (46) into Eq. (53), we obtain
\[ \sum_{k=1}^{N_2(m-1)} \sum_{n=0}^{N_2(n-i),k-1} \sum_{j=0}^{\max(k-2i-2)} h(-k+j)j_{m-n} \alpha_{n-i, j} \exp(-k\vartheta) \]
\[ = \chi_m \alpha_{m-1} \exp(-\vartheta) - \chi_{m-1}(\sigma - 1)g_{m-2}(\vartheta) + c_1 \exp(-\vartheta) \]
\[ + \sum_{k=1}^{N_2(m-1)} \sum_{n=0}^{N_2(n-i),k-1} \sum_{j=0}^{\max(k-2i-2)} h(-k+j)j_{m-n} \alpha_{n-i, j} \exp(-k\vartheta) \]
\[ \frac{1}{k(k^2 - 1)} + \sum_{k=1}^{N_2(m-1)} \sum_{n=0}^{N_2(n-i),k-1} \sum_{j=0}^{\max(k-2i-2)} h\alpha_{m-1,k} \exp(-k\vartheta) \] (54)
Collecting terms to the same order of \( \exp(-k\vartheta) \), we have
\[ \alpha_{m,1} = \chi_m \alpha_{m-1,1} - \chi_{m-1}(\sigma - 1)\alpha_{m-2,1} + c_1 \]
\[ \alpha_{m,k} = \frac{h}{k(k^2 - 1)} \sum_{n=0}^{N_2(n-i),k-1} \sum_{j=0}^{\max(k-2i-2)} (-k+j)j_{m-n} \alpha_{n-i, j} \]
\[ + \frac{h}{k} \chi_{m+2,1} \alpha_{m-1,1} + \chi_{m} \chi_{m+2,1} \sigma \alpha_{m-1,1} - \chi_{m-1} \chi_{m+2,1} (\sigma - 1) \alpha_{m-2,1} \] (55)
\[ \quad \text{for } k \geq 2 \]
where \( N_2(m) = 2m+2 \). The coefficient \( \gamma_m \) and the constant \( c_1 \) are determined by the two boundary conditions at \( \vartheta = 0 \) as
\[ r_m = \left[ \sum_{k=2}^{2m} (k^2 - 1) \alpha_{m-1,k} + \sum_{k=1}^{2m+2} \sum_{l=0}^{\min(2m-2i,2k-1)} \sum_{j=0}^{\max(2m-2i,2k-1)} \frac{j(j-k)}{k} \gamma_{m-n} \alpha_{n-i, j} \right]^{-1} \left[ \frac{1}{2} \alpha_{0,1}^2 + \frac{4}{3} \alpha_{0,1} \alpha_{0,2} + \alpha_{0,2}^2 \right] \] (56)
\[ c_1 = -h \sum_{k=2}^{2m+2} \sum_{l=0}^{\min(2m-2i,2k-1)} \sum_{j=0}^{\max(2m-2i,2k-1)} \frac{j(j-k)}{k^2 - 1} \gamma_{m-n} \alpha_{n-i, j} - h \sum_{k=2}^{2m+2} \alpha_{m-1,k} \] (57)
Thus, one can recursively calculate $x_{m,k}$ by using the first two terms $x_{0,1}$ and $x_{0,2}$, which are given in Eq. (45) as $x_{0,1} = -4/3$ and $x_{0,2} = 1/3$.

Eqs. (43) and (44) provide a family of solutions through the auxiliary parameters $h$ and $\sigma$. Fig. 7 shows the solution of $r$ as a function of $h$ and $\sigma$ at the fourth and fifth-order approximations. The exact solution has a value of $r = 3$ [13]. The series solution converges and the convergence region enlarges with increasing order of approximation. The fifth-order approximation provides highly accurate results with a large convergence region at the center of the plot. The optimal auxiliary parameters $h = -2$ and $\sigma = 1.3$ give an error $|r - 3|$ of $3.1 \times 10^{-4}$. The fifth-order approximation also gives highly accurate results of the loop soliton as shown in Fig. 8. In comparison, the homotopy analysis method with one auxiliary parameter does not provide a sufficiently accurate solution at the fifth-order and requires the 30th-order approximation to achieve the same accuracy [13].

Fig. 7. The $h$–$\sigma$ plot for the solution $r$. (a) Fourth-order. (b) Fifth-order.
Note here that the present series solution, Eqs. (43) and (46), has the same number of terms and uses the same base functions as the conventional approach with one auxiliary parameter [13]. The second auxiliary parameter $\mu$ only modifies the coefficients $a_{m,k}$ which in turn introduce a new dimension to the convergence characteristics of the solution. This allows convergence regions to exist at low orders of approximation and better convergence rates to be achieved at a given order. The results from both the KdV equation and the Vakhnenko equation show that the use of two auxiliary parameters improves the efficiency of the homotopy analysis method without increasing the complexity of the solutions.

5. Conclusions

The zero-order deformation equation with a single auxiliary parameter has remained the basis of the homotopy analysis method since its inception and provided analytical solutions to many nonlinear problems in science and engineering. In this paper, we develop a new form of the zero-order deformation equation with two auxiliary parameters and derive a general series solution for nonlinear differential equations. The second auxiliary parameter introduces a new dimension to the mapping of an initial approximation to the exact solution and improves both the convergence region and rate of the homotopy analysis method. Convergence theorems assure mapping of the initial approximation to the correct solution in the new homotopy.

The proposed method is illustrated through the KdV equation for periodic shallow-water waves and the Vakhnenko equation for one-loop soliton. The exact solutions allow a systematic examination of the accuracy and efficiency of the proposed series solution. The present approach reverts to the traditional homotopy analysis method in terms of convergence characteristics for weakly nonlinear problems. For strongly nonlinear problems, the second auxiliary parameter provides highly accurate series solution with fewer terms without increasing the complexity of the analytical solutions. The method described in this paper provides an alternative, effective analytical tool for nonlinear wave propagation problems.

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