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On the explicit, purely analytic solution of Von Kármán swirling viscous flow

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Abstract

A new analytic method for highly nonlinear problems, namely the homotopy analysis method, is applied to solve the Von Kármán swirling viscous flow, governed by a set of two fully coupled differential equations with strong nonlinearity. An explicit, purely analytic and uniformly valid solution is given, which agrees well with numerical results.

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1. Introduction

Von Kármán swirling viscous flow [1] is a famous classical problem in fluid mechanics. The original problem raised by Von Kármán is the viscous flow induced by an infinite rotating disk where the fluid far from the disk is at rest. Then the problem is generalized to include the case where the fluid itself is rotating as a solid body far from the disk with suction or injection at the disk surface. This introduces a parameter, i.e. the ratio of the angular velocity of the fluid at infinity to the angular velocity of the disk. Another generalization is to consider the viscous

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flow between two infinite coaxial rotating disks with suction or injection at both disks and this introduces another parameter, i.e. the Reynolds number determined by the distance of the two disks. All these problems are attacked, theoretically, numerically and experimentally, by many researchers such as Cochran [2], Fettis [3], Rogers and Lance [4], Benton [5], and so on (for details, please refer to Zandbergen and Dijkstra's review paper [6]). However, all of these results are either numerical or analytical-numerical.

In this paper, we focus on the original problem of Von Kármán [1]. Consider the steady, laminar, axially-symmetric viscous flow induced by an infinite disk rotating steadily with angular velocity Ω about the z-axis in a cylindrical coordinate system (r, θ, z). The motion of the incompressible viscous fluid, which is confined to the half-space $z > 0$ above the disk, is governed by the continuity and the exact Navier–Stokes equations

$$\frac{1}{r}\frac{\partial(rV_{\mathrm{r}})}{\partial r}+\frac{1}{r}\frac{\partial V_\theta}{\partial\theta}+\frac{\partial V_z}{\partial z}=0, \tag{1}$$

$$V_{\mathrm{r}}\frac{\partial V_{\mathrm{r}}}{\partial r}+V_z\frac{\partial V_{\mathrm{r}}}{\partial z}-\frac{V_\theta^2}{r}=v\left[\frac{\partial^2 V_{\mathrm{r}}}{\partial r^2}+\frac{1}{r}\frac{\partial V_{\mathrm{r}}}{\partial r}+\frac{\partial^2 V_{\mathrm{r}}}{\partial z^2}-\frac{V_{\mathrm{r}}}{r^2}\right]-\frac{1}{\rho}\frac{\partial\tilde{P}}{\partial r}, \tag{2}$$

$$V_{\mathrm{r}}\frac{\partial V_\theta}{\partial r}+V_z\frac{\partial V_\theta}{\partial z}+\frac{V_{\mathrm{r}}V_\theta}{r}=v\left[\frac{\partial^2 V_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial V_\theta}{\partial r}+\frac{\partial^2 V_\theta}{\partial z^2}-\frac{V_\theta}{r^2}\right], \tag{3}$$

$$V_{\mathrm{r}}\frac{\partial V_z}{\partial r}+V_z\frac{\partial V_z}{\partial z}=v\left[\frac{\partial^2 V_z}{\partial r^2}+\frac{1}{r}\frac{\partial V_z}{\partial r}+\frac{\partial^2 V_z}{\partial z^2}\right]-\frac{1}{\rho}\frac{\partial\tilde{P}}{\partial z}, \tag{4}$$

subject to the nonslip boundary conditions on the disk and boundary conditions at infinity

$$V_\theta=r\Omega,\quad V_{\mathrm{r}}=V_z=0\quad(z=0), \tag{5}$$

$$V_{\mathrm{r}}=V_z=0\quad(z=+\infty), \tag{6}$$

where ρ is the fluid density, v is the kinematic viscosity coefficient, $\tilde{P}$ is the pressure and V_r, V_θ, V_r are the velocity components in the radial, azimuthal and axial directions, respectively. Von Kármán [1] devised a similarity transformation

$$V_{\mathrm{r}}=(r\Omega)\ F(\eta),$$

$$V_\theta=(r\Omega)\ G(\eta),$$

$$V_z=\sqrt{v\Omega}\ H(\eta),$$

$$\tilde{P} = -\rho\nu\Omega\, P(\eta),$$

where

$$\eta = z\sqrt{\frac{\Omega}{\nu}}.$$

With this transformation he was able to reduce the governing partial differential Eqs. (1)–(4) to a set of ordinary differential equations

$$F'' = F^2 - G^2 + F'H, \tag{7}$$

$$G'' = G'H + 2FG, \tag{8}$$

$$HH' = P' + H'', \tag{9}$$

$$2F + H' = 0, \tag{10}$$

subject to the boundary conditions

$$F(0) = F(+\infty) = 0, \quad G(0) = 1, \quad G(+\infty) = 0, \quad H(0) = 0, \tag{11}$$

where the prime denotes the derivative with respect to η. According to (10) one has

$$F = -\frac{H'}{2}. \tag{12}$$

Substitute (12) into (7) and (8), one has

$$H''' - H''H + \frac{1}{2}H'H' - 2G^2 = 0, \tag{13}$$

$$G'' - HG' + H'G = 0, \tag{14}$$

with boundary conditions

$$H(0) = H'(0) = H'(+\infty) = 0, \quad G(0) = 1, \quad G(+\infty) = 0. \tag{15}$$

It should be emphasized that the above equations are deduced directly from the *exact* Navier–Stokes equations. This might be an important reason why this problem attracted so many researchers.

The above equations are fully coupled and highly nonlinear. Von Kármán [1] gave the approximate solution of these equations based on the momentum integral method. Cochran [2] pointed out the errors contained in Kármán's solution and used a kind of matching technique like Blasius' method to give a solution more accurate than Kármán's one. Fettis [3] devised a new asymptotic expansion which can describe the entire flow and Benton [5] gave an asymptotic solution better than Cochran's solution using Fettis's Method with only a trivial difference. However all the above-mentioned solutions employed some numerical methods and are essentially semi-analytic and semi-numerical ones.

Currently, Liao [7–11] proposed a new analytic method for highly nonlinear problems, namely the homotopy analysis method (HAM). Unlike perturbation techniques, the artificial small parameter method [12], the δ-expansion method [13] and Adomian's decomposition method [14], the homotopy analysis method *itself* provides us with a convenient way to *control* the convergence of approximation series and *adjust* convergence regions when necessary. Briefly speaking, the homotopy analysis method has the following advantages

1. it is valid even if a given nonlinear problem does *not* contain any small/large parameters *at all*;
2. it *itself* can provide us with a convenient way to *control* the convergence of approximation series and *adjust* convergence regions when necessary;
3. it can be employed to *efficiently* approximate a nonlinear problem by *choosing* different sets of base functions.

Besides, the homotopy analysis method logically contains Lyapunov's artificial small parameter method [12], the δ-expansion method [13], and Adomian's decomposition method [14] (as shown by Liao [7]).

The homotopy analysis method has been successfully applied to many nonlinear problems, such as nonlinear water waves [15], similarity boundary layer equations [16], Cheng-Chang equation [17], a third grade fluid past a porous plate [18], the flow of an Oldroyd 6-constant fluid [19], and so on. All of these verify the validity of the homotopy analysis method. In this paper we employ it to the original Von Kármán swirling viscous flow and to give an explicit, purely analytic, uniformly valid solution of above-mentioned fully coupled equations with strong nonlinearity.

2. The explicit analytic solution

Due to the boundary conditions (15), $H(\eta)$ and $G(\eta)$ can be expressed in form

$$H(\eta) = A_{0,0} + \sum_{i=1}^{+\infty}\sum_{j=0}^{+\infty} A_{i,j}\eta^m e^{-i\eta}, \tag{16}$$

$$G(\eta) = \sum_{i=1}^{+\infty}\sum_{j=0}^{+\infty} B_{i,j}\eta^m e^{-i\eta}, \tag{17}$$

respectively, where $A_{i,j}$ and $B_{i,j}$ are coefficients. They provide us with the *Rule of Solution Expression*, which plays an important role in the frame of the homotopy analysis method, as pointed out by Liao [8]. According to the boundary conditions (15) and the foregoing *Rule of Solution Expression* defined by (16) and (17), we choose

$$h_0(\eta) = -1 + e^{-\eta} + \eta e^{-\eta}, \tag{18}$$

$$g_0(\eta) = \mathrm{e}^{-\eta} \tag{19}$$

as the initial approximations of $H(\eta)$ and $G(\eta)$, and

$$\mathscr{L}_H(f) = f''' - f', \tag{20}$$

$$\mathscr{L}_G(f) = f'' - f \tag{21}$$

as our auxiliary linear operators, which have the following properties

$$\mathscr{L}_H[C_1 + C_2\mathrm{e}^{\eta} + C_3\mathrm{e}^{-\eta)}] = 0, \tag{22}$$

$$\mathscr{L}_G[C_4\mathrm{e}^{\eta} + C_5\mathrm{e}^{-\eta)}] = 0, \tag{23}$$

where C_1, C_2, C_3, C_4 and C_5 are constants.

Then, we construct the so-called zeroth-order deformation equations

$$(1-p)\ \mathscr{L}_H[\Lambda(\eta,p) - h_0(\eta)] = p\hbar_H\mathscr{N}_H[\Lambda(\eta,p),\Gamma(\eta,p)], \tag{24}$$

$$(1-p)\ \mathscr{L}_G[\Gamma(\eta,p) - g_0(\eta)] = p\hbar_G\mathscr{N}_G[\Lambda(\eta,p),\Gamma(\eta,p)], \tag{25}$$

subject to the boundary conditions

$$\Lambda(0,p) = \left.\frac{\partial\Lambda(\eta,p)}{\partial\eta}\right|_{\eta=0} = \left.\frac{\partial\Lambda(\eta,p)}{\partial\eta}\right|_{\eta=+\infty} = 0, \quad \Gamma(0,p) = 1, \quad \Gamma(+\infty,p) = 0, \tag{26}$$

where $\mathscr{N}_H$ and $\mathscr{N}_G$ are two nonlinear differential operators defined by

$$\mathscr{N}_H[\Lambda(\eta,p),\Gamma(\eta,p)] = \frac{\partial^3\Lambda(\eta,p)}{\partial\eta^3} - \Lambda(\eta,p)\frac{\partial^2\Lambda(\eta,p)}{\partial\eta^2} + \frac{1}{2}\left[\frac{\partial\Lambda(\eta,p)}{\partial\eta}\right]^2 - 2[\Gamma(\eta,p)]^2, \tag{27}$$

$$\mathscr{N}_G[\Lambda(\eta,p),\Gamma(\eta,p)] = \frac{\partial^2\Gamma(\eta,p)}{\partial\eta^2} - \Lambda(\eta,p)\frac{\partial\Gamma(\eta,p)}{\partial\eta} + \Gamma(\eta,p)\frac{\partial\Lambda(\eta,p)}{\partial\eta} \tag{28}$$

and $p \in [0,1]$ is the embedding parameter, $\hbar_H$ and $\hbar_G$ are auxiliary nonzero parameters. When $p = 0$, we get the solution

$$\Lambda(\eta,0) = h_0(\eta), \quad \Gamma(\eta,0) = g_0(\eta) \tag{29}$$

of Eqs. (24)–(26). When $p = 1$, Eqs. (24)–(26) are the same as the original Eqs. (13)–(15), respectively, so that

$$\Lambda(\eta,1) = H(\eta), \quad \Gamma(\eta,1) = G(\eta). \tag{30}$$

So, as p increases from 0 to 1, $\Lambda(\eta, p)$ varies from the initial guess $h_0(\eta)$ to the exact solution $H(\eta)$, so does $\Gamma(\eta, p)$ from $g_0(\eta)$ to $G(\eta)$. By Taylor's theorem and (29), we have

$$\Lambda(\eta,p) = h_0(\eta) + \sum_{k=1}^{+\infty} h_k(\eta)\ p^k, \tag{31}$$

$$\Gamma(\eta,p) = g_0(\eta) + \sum_{k=1}^{+\infty} g_k(\eta)\ p^k, \tag{32}$$

where

$$h_k(\eta) = \frac{1}{k!}\frac{\partial^k \Lambda(\eta;p)}{\partial p^k}\bigg|_{p=0}, \quad g_k(\eta) = \frac{1}{k!}\frac{\partial^k \Gamma(\eta;p)}{\partial p^k}\bigg|_{p=0}. \tag{33}$$

If the series (31) and (32) are convergent at $p = 1$, we have due to (30) that

$$H(\eta) = h_0(\eta) + \sum_{k=1}^{+\infty} h_k(\eta), \tag{34}$$

$$G(\eta) = g_0(\eta) + \sum_{k=1}^{+\infty} f_k(\eta). \tag{35}$$

Differentiating k times the zeroth-order deformation Eqs. (24)–(26) with respect to p and then dividing them by $k!$ and finally setting $p = 0$, we can get the kth-order deformation equations

$$\mathscr{L}_H[h_k(\eta) - \chi_k h_{k-1}(\eta)] = \hbar_H R_k^H(\eta), \tag{36}$$

$$\mathscr{L}_G[g_k(\eta) - \chi_k g_{k-1}(\eta)] = \hbar_G R_k^G(\eta), \tag{37}$$

subject to the boundary conditions

$$h_k(0) = h_k'(0) = h_k'(+\infty) = 0, \tag{38}$$

$$g_k(0) = g_k(+\infty) = 0, \tag{39}$$

where

$$R_k^H(\eta) = h_{k-1}'''(\eta) - \sum_{j=0}^{k-1}\left[h_j''(\eta)h_{k-1-j}(\eta) - \frac{1}{2}h_j'(\eta)h_{k-1-j}'(\eta) + 2g_j(\eta)g_{k-1-j}(\eta)\right], \tag{40}$$

$$R_k^G(\eta) = g_{k-1}''(\eta) - \sum_{j=0}^{k-1}\left[h_j(\eta)g_{k-1-j}'(\eta) - h_j'(\eta)g_{k-1-j}(\eta)\right], \tag{41}$$

and

$$\chi_k = \begin{cases} 0, & k \leqslant 1, \\ 1, & k > 1. \end{cases} \tag{42}$$

We apply the symbolic computation software MATHEMATICA to solve the linear Eqs. (36) and (39) successively in the order $k = 1, 2, 3, \ldots$, and we find that $h_k(\eta)$ and $g_k(\eta)$ can be expressed by

$$h_k(\eta) = \sum_{n=0}^{k+1} \mathrm{e}^{-n\eta} \sum_{i=0}^{k+1} \alpha_{k,n}^i \eta^i, \tag{43}$$

$$g_k(\eta) = \sum_{n=1}^{k+1} \mathrm{e}^{-n\eta} \sum_{i=0}^{k+1} \beta_{k,n}^i \eta^i. \tag{44}$$

We have got the explicit analytic solution

$$H(\eta) = \lim_{M\to+\infty} \sum_{k=0}^{M} \sum_{n=0}^{k+1} \mathrm{e}^{-n\eta} \sum_{i=0}^{k+1} \alpha_{k,n}^i \eta^i, \tag{45}$$

$$G(\eta) = \lim_{M\to+\infty} \sum_{k=0}^{M} \sum_{n=1}^{k+1} \mathrm{e}^{-n\eta} \sum_{i=0}^{k+1} \beta_{k,n}^i \eta^i \tag{46}$$

of the original Eqs. (13)–(15). Actually, we can calculate all the coefficients algebraically from the following first coefficients given by the initial guess

$$\alpha_{0,0}^0 = -1, \quad \alpha_{0,1}^0 = 1, \quad \alpha_{0,1}^1 = 1, \quad \beta_{0,1}^0 = 1. \tag{47}$$

At the Mth-order approximation, the solution can be expressed as follows

$$H(\eta) \approx \sum_{k=0}^{M} \sum_{n=0}^{k+1} \mathrm{e}^{-n\eta} \sum_{i=0}^{k+1} \alpha_{k,n}^i \eta^i, \tag{48}$$

$$G(\eta) \approx \sum_{k=0}^{M} \sum_{n=1}^{k+1} \mathrm{e}^{-n\eta} \sum_{i=0}^{k+1} \beta_{k,n}^i \eta^i. \tag{49}$$

Then, due to (9), it is easy to have

$$P(\eta) - P(0) = \frac{H^2(\eta)}{2} - H'(\eta). \tag{50}$$

And as mentioned before, $F(\eta)$ is given by (12).

3. The convergence of the solution

The explicit, analytic expressions (45) and (46) contain two auxiliary parameters $\hbar_H$ and $\hbar_G$. As pointed by Liao [7,8], the convergence region and rate of the approximations given by the homotopy analysis method are determined by the value of such kind of auxiliary parameters. Our calculations indicate that the series (45) and (46) converge to the numerical solution in the whole region of η, when

$$-1 \leqslant \hbar_H < 0, \quad -1 \leqslant \hbar_G < 0. \tag{51}$$

And when $\hbar_H = \hbar_G = -1$, the solution series converge fastest. The comparison of our analytic solution (48) and (49) with the numerical solution when $\hbar_H = \hbar_G = -1$ at different order of approximation are as shown in Figs. 1 and 2. Obviously, our analytic solution *uniformly* converges to the numerical results. Note that the series (49) of $G(\eta)$ converges faster than that of $H(\eta)$, mainly because the nonlinearity of the latter is stronger than the former.

The values of $H(+\infty)$ and

$$P(+\infty) - P(0) = \frac{H^2(+\infty)}{2}$$

have clear physical meanings. Cochran [2] gave $H(+\infty) = -0.886$ and $P(+\infty)-P(0) = 0.3925$. Fettis [3] obtained $H(+\infty) = -0.8840$ and $P(+\infty) - P(0) = 0.3907$. Benton's analytical-numerical value [5] (1966) of $H(+\infty)$ is -0.8845, which gives $P(+\infty) - P(0) = 0.3911$. Our pure analytic

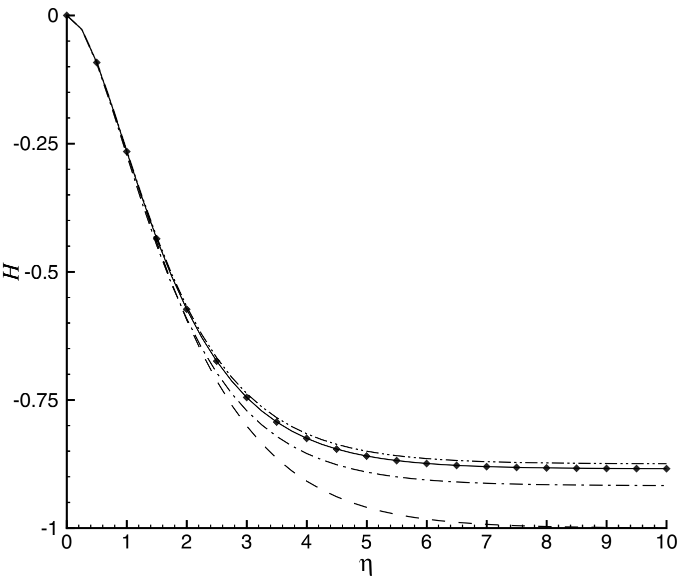

Fig. 1. Comparison of analytic approximation of $H(\eta)$ with numerical results. Symbol, numerical solution; dash line, initial guess; dash-dotted line, 5th-order approximation; dash-dot-dotted line, 10th-order approximation; solid line, 30th-order approximation.

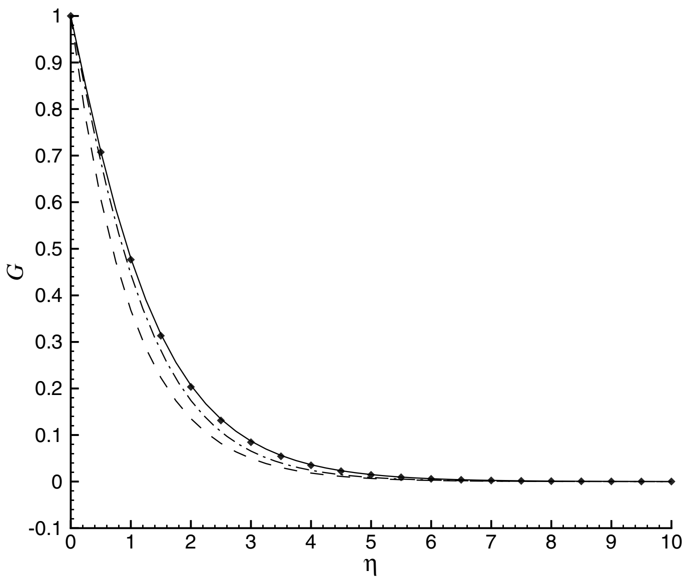

Fig. 2. Comparison of numerical results with analytic approximation of $G(\eta)$. Symbol, numerical solution; dash line, initial guess; dash-dotted line, 1st-order approximation; solid line, 10th-order approximation.

values of $H(+\infty)$ and $P(+\infty) - P(0)$ agree well with Benton's analytical-numerical ones [5] (1966) (as shown in Table 1).

Liao [7] proved in general such a convergence theorem, i.e. *any* a solution series given by the homotopy analysis method *must* be one of solutions of considered nonlinear problem. To show this for the Von Kármán swirling flow, let us consider the average residual errors of the two equations

$$E_H = \frac{1}{10}\int_0^{10}\left|H'''(\eta) - H''(\eta)H(\eta) + \frac{1}{2}H'(\eta)H'(\eta) - 2G(\eta)^2\right|\mathrm{d}\eta,$$

$$E_G = \frac{1}{10}\int_0^{10}|G''(\eta) - H(\eta)G'(\eta) + H'(\eta)G(\eta)|\,\mathrm{d}\eta.$$

Table 1
Approximations of $H(+\infty)$ and $P(+\infty) - P(0)$ when $\hbar_H = \hbar_G = -1$ at different order of approximation

Order of approximation	$H(+\infty)$	$P(+\infty) - P(0)$
0	−1	0.5
5	−0.9173	0.4207
10	−0.8747	0.3825
15	−0.8833	0.3901
20	−0.8845	0.3910
30	−0.8845	0.3911
40	−0.8845	0.3911
45	−0.8845	0.3911

Table 2
Residual errors of our analytic solutions when $\hbar_H = \hbar_G = -1$ at different order of approximation

Order of approximation	E_H	E_G
0	3.75×10^{-2}	5.00×10^{-2}
5	2.94×10^{-3}	2.66×10^{-3}
10	1.07×10^{-3}	5.31×10^{-4}
15	4.08×10^{-4}	1.05×10^{-4}
20	1.62×10^{-4}	2.25×10^{-5}
25	6.40×10^{-5}	1.04×10^{-5}
30	2.46×10^{-5}	7.95×10^{-6}
35	8.88×10^{-6}	5.47×10^{-6}
40	2.77×10^{-6}	3.33×10^{-6}
45	5.41×10^{-7}	1.87×10^{-6}

The average residual errors of series (48) and (49) at different order of approximations when $\hbar_H = \hbar_G = -1$ are listed in Table 2. Obviously, as the order M increases, the average residual errors decrease. This clearly indicates that our analytic solution series (45) and (46) are indeed the solution of the two original Eqs. (13) and (14).

4. Conclusion

In this paper, a new analytic method for highly nonlinear problems, namely the homotopy analysis method, is employed to give an explicit, totally analytic and uniformly valid solution of the famous Von Kármán swirling flow. Our analytic solution agrees well with numerical results (as shown in Figs. 1 and 2 and Tables 1 and 2).

We emphasize that our solution is *explicit* and *purely* analytic, i.e. the structure of the solution is known and the constant coefficients are given by recursive formula and it is *unnecessary* to use any numerical methods to get any coefficients. Note also that, different from equations governing Blasius and Falker–Skan boundary layer flows, the nonlinear Eqs. (13) and (14) are directly deduced from the *exact* Navier–Stokes equations without other assumptions.

Note that Eqs. (13) and (14) are fully coupled and highly nonlinear. This verifies that the homotopy analysis method is valid even for sets of fully coupled, highly nonlinear differential equations, and therefore can be applied to many other complicated nonlinear problems in science and engineering, especially in fluid mechanics which contains rich nonlinear phenomena.

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