Example 1: a system of ODEs in finite interval

Consider a system of coupled ODEs[9]

$$(1+K)f'''' - ReMf'' + 2Reff''' - Kg'' = 0, \tag{5}$$

$$(1+\frac{K}{2})g'' - ReK[2g - f''] + Re[2fg' - f'g] = 0, \tag{6}$$

$$\tag{7}$$

subject to

$$f(0) = 0,\ f(1) = 0,\ f'(1) = 1,\ f''(0) = 0, \tag{8}$$

$$g(1) = 0,\ g(0) = 0, \tag{9}$$

where K is the ratio of viscosities, Re is the Reynolds number and M is the Hartman number. Hayat[9] has solved this problem by the HAM.

Here we solve this problem by `BVPh 2.0`. Since there are two ODEs in system (5)–(6) without an unknown to be determined, we have `NumEQ = 2` and `TypeEQ=1`. The system is input as

```
TypeEQ = 1;
NumEQ = 2;
f[1,z_,{f_,g_},Lambda_]:=(1+K)*D[f,{z,4}]
    -Rey*M*D[f,{z,2}]+2*Rey*f*D[f,{z,3}]-K*D[g,{z,2}];
f[2,z_,{f_,g_},Lambda_]:=(1+K/2)*D[g,{z,2}]
    -Rey*K*(2*g-D[f,{z,2}])+Rey*(2*f*D[g,z]-D[f,z]*g);
```

The eight boundary conditions are defined as

```
NumBC = 6;
BC[1,z_,{f_,g_}]:=f/.z->0;
BC[2,z_,{f_,g_}]:=f/.z->1;
BC[3,z_,{f_,g_}]:=(D[f, z]-1)/.z->1;
BC[4,z_,{f_,g_}]:=D[f,{z,2}]/.z->0;
BC[5,z_,{f_,g_}]:=g/.z->1;
BC[6,z_,{f_,g_}]:=g/.z->0;
```

Now let us input the solution intervals

```
zL[1]=0;    zR[1]=1;
zL[2]=0;    zR[2]=1;
```

Since all the solution intervals are in finite intervals, we do not have to specify the integral interval to compute the squared residual.

The initial guesses are chosen as $f_0 = (z^3 - z)/2$ and $g_0 = 0$. They are input as

```
U[1,0] = (z^3-z)/2;
U[2,0] = 0;
```

The auxiliary linear operators are chosen as $\mathcal{L}_1 = \frac{\partial^4}{\partial z^4}$ and $\mathcal{L}_2 = \frac{\partial^2}{\partial z^2}$. They are defined as

```
L[1,u_]:=D[u,{z,4}];
L[2,u_]:=D[u,{z,2}];
```

Note that we use the delayed assignment `SetDelayed(:=)` to define these linear operators.

Without loss of generality, let us consider the case when $Re = M = 2$ and $K = 1/2$. These physical parameters are input as

```
Rey = M = 2;
K = 1/2;
```

At this time, we have input all the data for this problem, except the convergence-control parameters `c0[k]`. Hayat[9] chose the convergence-control parameters `c0[1]=c0[2]=-0.7` through $\hbar$-curve. Here we minimize the squared residual error of the 4th-order approximations to get optimal values for `c0[k]`

```
GetOptiVar[4,{},{c0[1],c0[2]}];
```

The convergence-control parameters `c0[1]` and `c0[2]` are found to be about -0.5825 and -0.721452 respectively.

Then we call the main module `BVPh` to get the 20th-order approximations

```
BVPh[1, 20];
```

The 20th-order approximations are stored in `U[i,20],i=1,2`, while the corresponding squared residual error is `ErrTotal[20]`. We can use

```
Plot[{U[1,20],U[2,20]},{z,0,1},AxesLabel->{"z",""},
PlotStyle->{{Thin, Red}, {Dashed, Blue}},
PlotRange->{{0, 1}, {-0.2, 0.2}}]
```

to plot the 20th-order approximations, which is shown in Fig. 3. This figure agrees with Hayat's[9] Fig. 9 and Fig. 12 when $M = 2$, $Re = 2$ and $K = 0.5$. The 20th-order approximations give the values of $f''(1) = 3.61076396287$ and $g'(1) = -0.738463496789$, which are the same with Hayat's result.[9] The total error of the system for every two order of approximations is plotted in Fig. 4 by the command

```
ListLogPlot[Table[{2 i, ErrTotal[2*i]}, {i, 1, 20}],
Joined -> True, Mesh -> All,
PlotRange -> {{2, 20}, {10^(-34), 1}},
AxesLabel -> {"m", "error"}]
```

We can see from it that the error decreases beautifully.

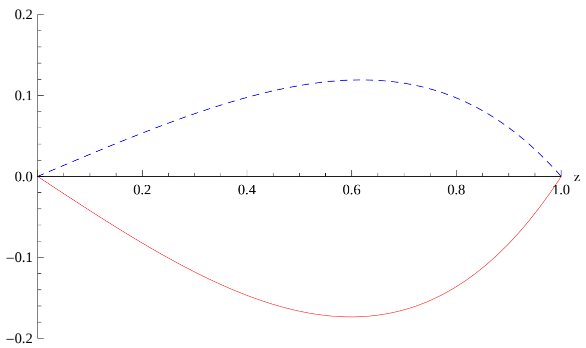

Fig. 3: The curve of $f(z)$ (solid), $g(z)$ (dashed) for Example 1.

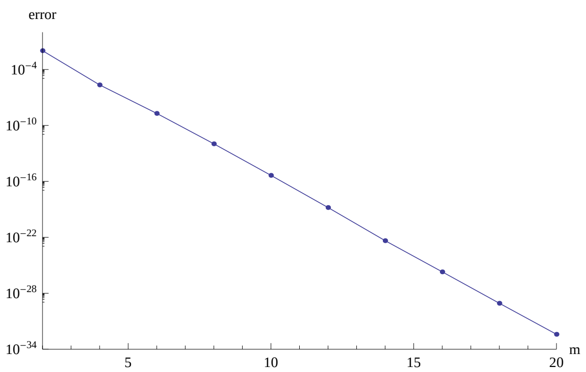

Fig. 4: Total error vs. order of approximation for Example 1.

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