Basic Ideas and Brief History of the Homotopy Analysis Method

1 Introduction

Nonlinear equations are much more difficult to solve than linear ones, especially by means of analytic methods. In general, there are two standards for a satisfactory analytic method of nonlinear equations:

(a) It can *always* provide analytic approximations *efficiently*.

(b) It can ensure that analytic approximations are *accurate* enough for *all* physical parameters.

Using above two standards as criterion, let us compare different analytic techniques for nonlinear problems.

Perturbation techniques [11, 15, 30, 37, 38, 43] are widely applied in science and engineering. Perturbation techniques are based on small/large physical parameters (called perturbation quantities) in governing equations or initial/boundary conditions. In general, perturbation approximations are expressed in series of perturbation quantities, and a nonlinear equation is replaced by an infinite number of linear (sometimes even nonlinear) sub-problems, which are completely determined by the original governing equation and especially by the place where perturbation quantities appear. Perturbation methods are simple, and easy to understand. Especially, based on small physical parameters, perturbation approximations often have clear physical meanings. Unfortunately, *not* every nonlinear problem has such kind of perturbation quantity. Besides, even if there exists such a small physical parameter, the sub-problem might have no solutions, or might be so complicated that only a few sub-problems can be solved. Thus, it is *not* guaranteed that one can *always* gain perturbation approximations for any a given nonlinear problem. More importantly, it is well-known that most perturbation approximations are valid only for small physical parameters. So, it is *not* guaranteed that a perturbation result is valid in the whole region of *all* physical parameters. Thus, perturbation techniques do not satisfy not only the standard (a) but also the standard (b) mentioned above.

To overcome the restrictions of perturbation techniques, some traditional non-perturbation methods are developed, such as Lyapunov's artificial small parameter method [31], the δ-expansion method [9, 14], Adomian decomposition method [4–7, 10, 41], and so on. In principle, all of these methods are based on a so-called artificial parameter, and approximation solutions are expanded into series of such kind of artificial parameter. The artificial small parameter is often used in such a way that one can always gain approximation solutions for any a given nonlinear equation. Compared with perturbation techniques, this is indeed a great progress. However, in theory, one can put the artificial small parameter in many different ways, but unfortunately there are no theories to guide us how to put it in a better place so as to gain a better approximation. For example, Adomian decomposition method simply uses the linear operator d^k/dx^k in most cases, where k is the highest order of derivative of governing equations, and therefore it is rather easy to gain solutions of the corresponding sub-problems by means of integrating k times with respect to x. However, such simple linear operator gives approximations in power-series, which unfortunately has often a finite radius of convergence. Thus, Adomian decomposition method can not ensure the convergence of its approximation series. Generally speaking, all traditional non-perturbation methods, such as Lyapunov's artificial small parameter method [31], the δ-expansion method [9, 14] and Adomian decomposition method [4–7, 10, 41], can *not* guarantee the convergence of approximation series. So, these traditional non-perturbation methods satisfy only the standard (a) but *not* the standard (b) mentioned before.

2 The early HAM

In 1992 Liao [17] took the lead to apply the homotopy [13], a basic concept in topology [42], to gain analytic approximations of nonlinear differential equations. The early homotopy analysis method (HAM) was first described by Liao [17] in his PhD dissertation in 1992. For a given nonlinear differential equation

$$\mathcal{N}[u(x)] = 0, \quad x \in \Omega,$$

where $\mathcal{N}$ is a nonlinear operator and $u(x)$ is a unknown function, Liao [17] constructed a *one-parameter* family of equations in the embedding parameter $q \in [0, 1]$, called the zeroth-order deformation equation

$$(1 - q)\mathcal{L}\left[\phi(x; q) - u_0(x)\right] + q\,\mathcal{N}[\phi(x; q)] = 0, \quad x \in \Omega,\ q \in [0, 1], \tag{1}$$

where $\mathcal{L}$ is an auxiliary linear operator and $u_0(x)$ is an initial guess. In theory, the concept of homotopy in topology provides us much larger freedom to choose both of the auxiliary linear operator $\mathcal{L}$ and the initial guess than the traditional non-perturbation methods mentioned above, as pointed out by Liao [19, 20, 23, 29]. At $q = 0$ and $q = 1$, we have $\phi(x; 0) = u_0(x)$ and $\phi(x; 1) = u(x)$, respectively. So, as the embedding parameter $q \in [0, 1]$ increases from 0 to 1, the solution $\phi(x; q)$ of the zeroth-order deformation equations varies (or deforms) from the initial guess $u_0(x)$ to the exact solution $u(x)$ of the original nonlinear differential equation $\mathcal{N}[u(x)] = 0$. Such kind of continuous variation is called deformation in topology, and this is the reason why we call (1) the zeroth-order deformation equation. Since $\phi(x; q)$ is also dependent upon the embedding parameter $q \in [0, 1]$, we can expand it into the Maclaurin series with respect to q:

$$\phi(x; q) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x)\, q^n, \tag{2}$$

called the homotopy-Maclaurin series. Note that we have extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$. Assuming that, the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$ are so properly chosen that the above homotopy-Maclaurin series converges at $q = 1$, we have the so-called homotopy-series solution

$$u(x) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x), \tag{3}$$

which satisfies the original equation $\mathcal{N}[u(x)] = 0$, as proved by Liao [19,20] in general. Here, $u_n(x)$ is governed by the so-called high-order deformation equation

$$\mathcal{L}\left[u_n(x) - \chi_n\, u_{n-1}(x)\right] = -\delta_{n-1}(x), \tag{4}$$

where χ_k equals to 1 when $k \geq 2$ but zero otherwise, and

$$\delta_k(x) = \left\{ \frac{1}{k!} \frac{\partial^k \mathcal{N}[\phi(x;q)]}{\partial q^k} \right\} \bigg|_{q=0}. \tag{5}$$

The high-order deformation equation (4) is always linear with the known term on the right-hand side, therefore is easy to solve, as long as we choose the auxiliary linear operator $\mathcal{L}$ properly.

3 The normal HAM

Unfortunately, Liao [18,20] found that the early HAM mentioned above can not always guarantee the convergence of approximation series of nonlinear equations in general. To overcome this restriction, Liao [18] in 1997 introduced such a non-zero auxiliary parameter c_0 to construct a *two-parameter* family of equations*, i.e. the zeroth-order deformation equation

$$(1-q)\mathcal{L}\left[\phi(x;q) - u_0(x)\right] = c_0\, q\, \mathcal{N}[\phi(x;q)], \quad x \in \Omega,\ q \in [0,1]. \tag{6}$$

The corresponding high-order deformation equation reads

$$\mathcal{L}\left[u_n(x) - \chi_n\, u_{n-1}(x)\right] = c_0\, \delta_{n-1}(x), \tag{7}$$

where $\delta_k(x)$ is defined by (5). In this way, the homotopy-series solution (3) is not only dependent upon the physical variable x but also the auxiliary parameter c_0. Mathematically, it was found [18–20] that the auxiliary parameter c_0 can adjust and control the convergence region and rate of homotopy-series solutions, although c_0 has no physical meanings at all. For detailed mathematical proofs, please refer to Chapter 5 of [29]. In essence, the use of the auxiliary parameter c_0 introduces us

*Liao [18] originally used the symbol $\hbar$ to denote the non-zero auxiliary parameter. But, $\hbar$ is well-known as Planck's constant in quantum mechanics. To avoid misunderstanding, $\hbar$ is replaced by the symbol c_0 in the book [29], which denotes the "basic" convergence-control parameter.

one more "artificial" degree of freedom, which greatly improves the early HAM: it is the auxiliary parameter c_0 which provides us a convenient way to guarantee the convergence of homotopy-series solution. For example, Liang & Jeffrey [16] illustrated that, when analytic approximations given by the other analytic method is divergent in the whole domain, one can gain convergent series solution simply by choosing a proper auxiliary parameter c_0. This is the reason why we call c_0 the *convergence-control parameter*.

The use of the convergence-control parameter c_0 is indeed a great progress in the frame of the HAM. It seems that more "artificial" degrees of freedom imply larger possibility to gain better approximations by means of the homotopy analysis method. Thus, Liao [19] in 1999 further introduced more "artificial" degrees of freedom by using the zeroth-order deformation equation in a more general form:

$$[1-\alpha(q)]\mathcal{L}\left[\phi(x;q)-u_0(x)\right] = c_0\,\beta(q)\,\mathcal{N}[\phi(x;q)], \quad x\in\Omega,\ q\in[0,1], \tag{8}$$

where $\alpha(q)$ and $\beta(q)$ are the so-called *deformation functions*† satisfying

$$\alpha(0)=\beta(0)=0, \alpha(1)=\beta(1)=1, \tag{9}$$

whose Taylor series

$$\alpha(q)=\sum_{m=1}^{+\infty}\alpha_m\,q^m, \quad \beta(q)=\sum_{m=1}^{+\infty}\beta_m\,q^m, \tag{10}$$

are convergent for $|q|\leq 1$. The corresponding high-order deformation equation reads

$$\mathcal{L}\left[u_m(x)-\sum_{k=1}^{m-1}\alpha_k\,u_{m-k}(x)\right] = c_0\sum_{k=1}^{m}\beta_k\,\delta_{m-k}(x), \tag{11}$$

where $\delta_k(x)$ is defined by (5).

In fact, the zeroth-order deformation equation (8) can be further generalized, as shown by Liao [20,21,24]. Obviously, there are an infinite number of the deformation functions as defined above. Thus, the approximation series given by the HAM can contain so many "artificial" degrees of freedom that they provide us great possibility to guarantee the convergence of homotopy-series solution. Note that $u_n(x)$ is always governed by the same auxiliary linear operator $\mathcal{L}$, and we have great freedom to choose $\mathcal{L}$ in such a way that $u_n(x)$ is easy to obtain. More importantly, for given auxiliary linear operator $\mathcal{L}$ and initial guess, we can always gain convergent homotopy-series solution by choosing proper convergence-control parameter c_0 and proper deformation functions $\alpha(q)$ and $\beta(q)$. Inversely, the guarantee of the convergence of homotopy-series solutions also provides us freedom to choose the auxiliary linear operator $\mathcal{L}$ and initial guess. It is due to such kind of guarantee in the frame of the HAM that a

$^\dagger\alpha(q)$ and $\beta(q)$ were called "approaching function" in some early articles about the homotopy-analysis method. In the book [29], they are defined as "deformation function", which better reveals its relationship with the zeroth-order deformation equations nonlinear ODE with variable coefficients can be transferred into a sequence of linear ODEs with constant coefficients [26], that a nonlinear PDE can be transferred into an infinite number of linear ODEs [22, 25], that several coupled nonlinear ODEs can be transferred into an infinite number of linear decoupled ODEs [45], and that even a 2nd-order nonlinear PDE can be replaced by an infinite number of 4th-order linear PDEs [23]. In fact, it is such kind of guarantee for convergence of series solutions, together with the extremely large freedom in choice of the auxiliary linear operators, that greatly simplifies finding convergent series of nonlinear equations in the frame of the HAM, as illustrated in above-mentioned articles [22, 23, 25, 26, 45]. On the other hand, without such kind of guarantee of convergence, we have in practice no *true* freedom to choose the auxiliary linear operator $\mathcal{L}$, because the freedom to get a divergent series solution has no meanings at all! For example, Liang & Jeffrey [16] pointed out that the series solution given by means of the so-called "homotopy perturbation method¨ [12] is divergent at all points except the initial guess, and thus has completely no scientific meanings. So, unlike perturbation techniques and the traditional non-perturbation methods mentioned above, the HAM satisfies both the standard (a) and (b).

How to find a proper convergence-control parameter c_0 so as to gain a convergent series solution? A straight-forward way to check the convergence of a homotopy-series solution is to substitute it into original governing equations and boundary/initial conditions and then to check the corresponding squared residual integrated in the whole region. However, when the approximations contain unknown convergence-control parameters and/or other unknown physical parameters, it is time-consuming to calculate the squared residual at high-order of approximations. To avoid the time-consuming computation, Liao [18–20] suggested to investigate the convergence of some special quantities which often have important physical meanings. For example, one can consider the convergence of $u'(0)$ and $u''(0)$ of a nonlinear differential equation $\mathcal{N}[u(x)] = 0$. It is found by Liao [18–20] that there often exists such an effective-region $\mathbf{R}_c$ that any $c_0 \in \mathbf{R}_c$ gives a convergent series solution of such kind of quantities. Besides, such kind of effective-region can be found, although approximately, by plotting the curves of these unknown quantities versus c_0. For example, for a nonlinear differential equation $\mathcal{N}[u(x)] = 0$, one may approximately determine $\mathbf{R}_c$ by plotting curves $u'(0) \sim c_0$, $u''(0) \sim c_0$ and so on. These curves are called "c_0-curves" or "curves for convergence-control parameter"‡, which have been successfully applied to solve many nonlinear problems [20].

4 The optimal HAM

However, such kind of c_0-curves can not tell us the best convergence-control parameter c_0, which corresponds to the fastest convergent series. In 2007, Yabushita, Yamashita and Tsuboi [44] applied the HAM to solve two coupled nonlinear ODEs. They suggested the so-called "optimization method" to find out the two optimal convergence-

‡The c_0-curve was originally called the $\hbar$-curve, and $\mathbf{R}_c$ was originally denoted by $\mathbf{R}_\hbar$.

control parameters by means of the minimum of the squared residual of governing equations. Let

$$E_m = \int_\Omega \left\{ \mathcal{N}\left[\sum_{n=1}^{m} u_n(x) \right] \right\}^2 d\Omega$$

denote the squared residual of the mth-order approximation of the governing equation $\mathcal{N}(u) = 0$, integrated in the whole domain Ω. In theory, if the squared residual E_m tends to zero, then $\sum_{n=0}^{+\infty} u_n(x)$ is a series solution of the original equation $\mathcal{N}(u) = 0$. So, if there exists only one convergence-control parameter c_0, the so-called effective-region $\mathbf{R}_c$ of the convergence-control parameter c_0 is defined by

$$\mathbf{R}_c = \left\{ c_0 \;\middle|\; \lim_{m\to+\infty} E_m(c_0) = 0 \right\}.$$

Besides, at the given order of approximation, the minimum of the squared residual E_m corresponds to the optimal approximation. So, the curves of the squared residual E_m versus c_0 indicate not only the effective-region $\mathbf{R}_c$ of the convergence-control parameter c_0, but also the optimal value of c_0 that corresponds to the minimum of E_m. Note that one can gain the squared residual of an equation at any order of approximations, even if the exact solutions are unknown. Therefore, it is a very good idea of Yabushita, Yamashita and Tsuboi [44] to use the squared residual to find out the effective-region $\mathbf{R}_c$ and the optimal convergence-control parameters.

In 2008, Akyildiz and Vajravelu [8] gained optimal convergence-control parameter by the minimum of squared residual of governing equation, and found that the corresponding homotopy-series solution converges very quickly.

In 2008, Marinca and Herişanu [32] combined c_0 and $\beta(q)$ in the zeroth-order deformation equation (8) as one function $\check{\beta}(q)$ with $\check{\beta}(0) = 0$ but $\check{\beta}(1) \neq 1$, and considered such a family of equations

$$(1-q)\mathcal{L}\left[\phi(x;q) - u_0(x)\right] = \check{\beta}(q)\, \mathcal{N}[\phi(x;q)], \quad q \in [0,1], \tag{12}$$

where the Taylor series

$$\check{\beta}(q) = \sum_{n=1}^{+\infty} c_n\, q^n$$

converges at $q = 1$. The above equation is a special case of (8), if we choose

$$\alpha(q) = q, \qquad \beta(q) = \frac{1}{c_0}\sum_{n=1}^{+\infty} c_n\, q^n = \frac{\check{\beta}(q)}{c_0}, \qquad c_0 = \sum_{n=1}^{+\infty} c_n \neq 0. \tag{13}$$

So, the so-called "optimal homotopy asymptotic method" [32, 33] is still in the frame of the HAM. Even so, Marinca and Herişanu's approach [32] is interesting, which has the advantage that $\check{\beta}(1) = 1$ is *unnecessary* so that we have *larger* freedom to choose the auxiliary parameters c_n: all of them become the so-called convergence-control parameters. Marinca and Herişanu [32] developed the so-called "optimal homotopy asymptotic method" by minimizing the squared residual E_m: at the mth-order of approximation, a set of nonlinear algebraic equations about $c_1, c_2, \cdots, c_m$ are solved so as to find their optimal values. In theory, the more the convergence-control parameters are used, the better approximation we should obtain by this optimal HAM. However, with too many unknown parameters, it is time-consuming to find out the optimal convergence-control parameters, especially at high-order of approximations for a complicated nonlinear problem. For example, Niu and Wang [39] illustrated that the optimal approach given by Marinca et al. [32,33] is time-consuming [34,40], although their optimal HAM [32] is more rigorous in theory than Nou and Wang's ones. It seems that one had to balance the rigorousness in theory against the computational efficiency in practice.

To increase the computational efficiency, Liao [28] developed in 2010 an optimal HAM with only three convergence-control parameters. Like Marinca and Herişanu's approach [32, 33], this optimal HAM is also based on the zeroth-order deformation equation (8). However, two types of special deformation-functions are used, which are determined completely by only one characteristic parameter $|c_1| < 1$ and $|c_2| < 1$, respectively. In this way, there exist at most only three unknown convergence-control parameters c_0, c_1 and c_2 at *any* order of approximations. In addition, the discrete squared residual is first introduced by Liao [28] to efficiently find out the optimal convergence-control parameters.

In 2012, Liao [29] suggested a generalized optimal HAM by choosing $\alpha(q) = q$ and such a special deformation-function

$$\beta(q) = \frac{1}{c_0} \sum_{n=1}^{\kappa} c_n \, q^n$$

in (8), where $\kappa \geq 1$ is a positive integer and

$$c_0 = \sum_{n=1}^{\kappa} c_n \neq 0.$$

The corresponding zeroth-order deformation equation (8) reads

$$(1-q)\mathcal{L}[\phi(x;q) - u_0(x)] = \left(\sum_{n=1}^{\kappa} c_n \, q^n \right) \mathcal{N}\left[\phi(x;q)\right],$$

and the corresponding mth-order deformation equation becomes

$$\mathcal{L}[u_m(x) - \chi_m \, u_{m-1}(x)] = \sum_{n=1}^{\min\{m,\kappa\}} c_n \, \delta_{m-n}(x),$$

where $\delta_k(x)$ is defined by (5). Note that the mth-order homotopy-approximation

$$u(x) \sim u_0(x) + \sum_{n=1}^{m} u_n(x)$$

contains at most the κ unknown convergence-control parameters

$$c_1, c_2, c_3, \cdots, c_\kappa.$$

Therefore, in theory, there exist a *finite* number of unknown convergence-control parameters

$$c_1, c_2, c_3, \cdots, c_\kappa$$

even as $m \to +\infty$. In this case, the optimal mth-order homotopy-approximation is given by a set of $\min\{m, \kappa\}$ nonlinear algebraic equations

$$\frac{\partial E_m}{\partial c_n} = 0, \qquad 1 \leq n \leq \min\{m, \kappa\}. \tag{14}$$

The above optimal HAM becomes exactly the so-called "optimal homotopy asymptotic method" suggested by Marinca and Herişanu [32], if $\kappa \to \infty$. Besides, when $c_1 = c_0$ and $c_n = 0$ for $n > 1$, it becomes the basic optimal HAM. Therefore, this optimal HAM is more general. For details, please refer to the book [29].

5 Mathematica package BVPh

Inspirited by the general validity of the HAM in so many different fields and by the ability of "computing with functions instead of numbers" provided by computer algebra system such as Mathematica and Maple, a HAM-based Mathematica package **BVPh** (version 1.0) is developed for highly nonlinear boundary-value/eigenvalue equations. The **BVPh** 1.0 is mainly valid for nonlinear ordinary differential equation with singularity, multiple solutions and/or multi-point boundary conditions in a finite or an infinite interval. It is even valid for some nonlinear partial differential equations related to boundary-layer flows. The aim is to develop a kind of analytic tool for as *many* nonlinear boundary-value problems (BVPs) as *possible* such that multiple solutions of some highly nonlinear BVPs can be conveniently found out, and that the infinite interval and singularity of governing equations and/or multi-point boundary conditions can be easily resolved.

Twelve examples for the use of the **BVPh** 1.0 are given in Part II of the book [29]. As an open resource, the **BVPh** 1.0 with a simple user's guide is free available at

http://numericaltank.sjtu.edu.cn/BVPh.htm.

The higher version of the **BVPh** will be issued in future.

6 Some other approaches based on the HAM

6.1 The specctral HAM

In 2010, Motsa et al. [35, 36] suggested the so-called "spectral homotopy analysis method" (SHAM) by using the Chebyshev pseudo-spectral method to solve the linear high-order deformation equations. Since the SHAM combines the HAM with the numerical techniques, it provides us larger freedom to choose auxiliary linear operators. Thus, one can choose more complicated auxiliary linear operators in the frame of the SHAM.

In theory, any a continuous function in a bounded interval can be best approximated by Chebyshev polynomial. So, the SHAM provides larger freedom to choose the auxiliary linear operator $\mathcal{L}$ and initial guess. The basic idea of the SHAM might be expanded to solve nonlinear partial differential equations. Besides, it is easy to employ the optimal convergence-control parameter in the frame of the SHAM. Thus, the SHAM has great potential to solve more complicated nonlinear problems in science and engineering, although further modifications in theory and more applications are needed.

Chebyshev polynomial is a kind of special function. There are many other special functions such as Hermite polynomial, Legendre polynomial, Airy function, Bessel function, Riemann zeta function, hypergeometric functions and so on. Since the HAM provides us extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess, it should be possible to develop a "generalized spectral HAM" which can use a proper special function for a given nonlinear problem.

6.2 The predictor HAM

Abbasbandy et al. [1–3] proposed the so-called "the predictor homotopy analysis method" (PHAM) to predict the multiplicity of solutions of nonlinear equations. Using the PHAM, they obtained multiple solutions of some nonlinear differential equations by means of different values of the convergence-control parameter c_0 with the **same** auxiliary linear operator $\mathcal{L}$ and even the **same** initial guess. As pointed out by Abbasbandy et al. [2], this trait makes HAM to be different from the other analytical techniques which are used to approach one solution but possibly lose the others.

For details, please refer to Abbasbandy et al. [1–3].

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