2 The early HAM

In 1992 Liao [17] took the lead to apply the homotopy [13], a basic concept in topology [42], to gain analytic approximations of nonlinear differential equations. The early homotopy analysis method (HAM) was first described by Liao [17] in his PhD dissertation in 1992. For a given nonlinear differential equation

$$\mathcal{N}[u(x)] = 0, \quad x \in \Omega,$$

where $\mathcal{N}$ is a nonlinear operator and $u(x)$ is a unknown function, Liao [17] constructed a *one-parameter* family of equations in the embedding parameter $q \in [0,1]$, called the zeroth-order deformation equation

$$(1-q)\mathcal{L}\left[\phi(x;q) - u_0(x)\right] + q\,\mathcal{N}[\phi(x;q)] = 0, \quad x \in \Omega,\; q \in [0,1], \tag{1}$$

where $\mathcal{L}$ is an auxiliary linear operator and $u_0(x)$ is an initial guess. In theory, the concept of homotopy in topology provides us much larger freedom to choose both of the auxiliary linear operator $\mathcal{L}$ and the initial guess than the traditional non-perturbation methods mentioned above, as pointed out by Liao [19, 20, 23, 29]. At $q = 0$ and $q = 1$, we have $\phi(x;0) = u_0(x)$ and $\phi(x;1) = u(x)$, respectively. So, as the embedding parameter $q \in [0,1]$ increases from 0 to 1, the solution $\phi(x;q)$ of the zeroth-order deformation equations varies (or deforms) from the initial guess $u_0(x)$ to the exact solution $u(x)$ of the original nonlinear differential equation $\mathcal{N}[u(x)] = 0$. Such kind of continuous variation is called deformation in topology, and this is the reason why we call (1) the zeroth-order deformation equation. Since $\phi(x;q)$ is also dependent upon the embedding parameter $q \in [0,1]$, we can expand it into the Maclaurin series with respect to q:

$$\phi(x;q) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x)\, q^n, \tag{2}$$

called the homotopy-Maclaurin series. Note that we have extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$. Assuming that, the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$ are so properly chosen that the above homotopy-Maclaurin series converges at $q = 1$, we have the so-called homotopy-series solution

$$u(x) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x), \tag{3}$$

which satisfies the original equation $\mathcal{N}[u(x)] = 0$, as proved by Liao [19,20] in general. Here, $u_n(x)$ is governed by the so-called high-order deformation equation

$$\mathcal{L}\left[u_n(x) - \chi_n\, u_{n-1}(x)\right] = -\delta_{n-1}(x), \tag{4}$$

where χ_k equals to 1 when $k \geq 2$ but zero otherwise, and

$$\delta_k(x) = \left\{ \frac{1}{k!} \frac{\partial^k \mathcal{N}[\phi(x;q)]}{\partial q^k} \right\}\Bigg|_{q=0}. \tag{5}$$

The high-order deformation equation (4) is always linear with the known term on the right-hand side, therefore is easy to solve, as long as we choose the auxiliary linear operator $\mathcal{L}$ properly.