4 The optimal HAM

However, such kind of c_0-curves can not tell us the best convergence-control parameter c_0, which corresponds to the fastest convergent series. In 2007, Yabushita, Yamashita and Tsuboi [44] applied the HAM to solve two coupled nonlinear ODEs. They suggested the so-called "optimization method" to find out the two optimal convergence-control parameters by means of the minimum of the squared residual of governing equations. Let

$$E_m = \int_{\Omega} \left\{ \mathcal{N}\left[\sum_{n=1}^{m} u_n(x) \right] \right\}^2 d\Omega$$

denote the squared residual of the mth-order approximation of the governing equation $\mathcal{N}(u) = 0$, integrated in the whole domain Ω. In theory, if the squared residual E_m tends to zero, then $\sum_{n=0}^{+\infty} u_n(x)$ is a series solution of the original equation $\mathcal{N}(u) = 0$. So, if there exists only one convergence-control parameter c_0, the so-called effective-region $\mathbf{R}_c$ of the convergence-control parameter c_0 is defined by

$$\mathbf{R}_c = \left\{ c_0 \;\middle|\; \lim_{m\to+\infty} E_m(c_0) = 0 \right\}.$$

Besides, at the given order of approximation, the minimum of the squared residual E_m corresponds to the optimal approximation. So, the curves of the squared residual E_m versus c_0 indicate not only the effective-region $\mathbf{R}_c$ of the convergence-control parameter c_0, but also the optimal value of c_0 that corresponds to the minimum of E_m. Note that one can gain the squared residual of an equation at any order of approximations, even if the exact solutions are unknown. Therefore, it is a very good idea of Yabushita, Yamashita and Tsuboi [44] to use the squared residual to find out the effective-region $\mathbf{R}_c$ and the optimal convergence-control parameters.

In 2008, Akyildiz and Vajravelu [8] gained optimal convergence-control parameter by the minimum of squared residual of governing equation, and found that the corresponding homotopy-series solution converges very quickly.

In 2008, Marinca and Herişanu [32] combined c_0 and $\beta(q)$ in the zeroth-order deformation equation (8) as one function $\check{\beta}(q)$ with $\check{\beta}(0) = 0$ but $\check{\beta}(1) \neq 1$, and considered such a family of equations

$$(1-q)\mathcal{L}\left[\phi(x;q) - u_0(x)\right] = \check{\beta}(q)\,\mathcal{N}[\phi(x;q)], \quad q \in [0,1], \tag{12}$$

where the Taylor series

$$\check{\beta}(q) = \sum_{n=1}^{+\infty} c_n\, q^n$$

converges at $q = 1$. The above equation is a special case of (8), if we choose

$$\alpha(q) = q, \quad \beta(q) = \frac{1}{c_0}\sum_{n=1}^{+\infty} c_n\, q^n = \frac{\check{\beta}(q)}{c_0}, \quad c_0 = \sum_{n=1}^{+\infty} c_n \neq 0. \tag{13}$$

So, the so-called "optimal homotopy asymptotic method" [32, 33] is still in the frame of the HAM. Even so, Marinca and Herişanu's approach [32] is interesting, which has the advantage that $\breve{\beta}(1) = 1$ is *unnecessary* so that we have *larger* freedom to choose the auxiliary parameters c_n: all of them become the so-called convergence-control parameters. Marinca and Herişanu [32] developed the so-called "optimal homotopy asymptotic method" by minimizing the squared residual E_m: at the mth-order of approximation, a set of nonlinear algebraic equations about $c_1, c_2, \cdots, c_m$ are solved so as to find their optimal values. In theory, the more the convergence-control parameters are used, the better approximation we should obtain by this optimal HAM. However, with too many unknown parameters, it is time-consuming to find out the optimal convergence-control parameters, especially at high-order of approximations for a complicated nonlinear problem. For example, Niu and Wang [39] illustrated that the optimal approach given by Marinca et al. [32, 33] is time-consuming [34, 40], although their optimal HAM [32] is more rigorous in theory than Nou and Wang's ones. It seems that one had to balance the rigorousness in theory against the computational efficiency in practice.

To increase the computational efficiency, Liao [28] developed in 2010 an optimal HAM with only three convergence-control parameters. Like Marinca and Herişanu's approach [32, 33], this optimal HAM is also based on the zeroth-order deformation equation (8). However, two types of special deformation-functions are used, which are determined completely by only one characteristic parameter $|c_1| < 1$ and $|c_2| < 1$, respectively. In this way, there exist at most only three unknown convergence-control parameters c_0, c_1 and c_2 at *any* order of approximations. In addition, the discrete squared residual is first introduced by Liao [28] to efficiently find out the optimal convergence-control parameters.

In 2012, Liao [29] suggested a generalized optimal HAM by choosing $\alpha(q) = q$ and such a special deformation-function

$$\beta(q) = \frac{1}{c_0} \sum_{n=1}^{\kappa} c_n \, q^n$$

in (8), where $\kappa \geq 1$ is a positive integer and

$$c_0 = \sum_{n=1}^{\kappa} c_n \neq 0.$$

The corresponding zeroth-order deformation equation (8) reads

$$(1-q)\mathcal{L}[\phi(x;q) - u_0(x)] = \left(\sum_{n=1}^{\kappa} c_n \, q^n \right) \mathcal{N}\left[\phi(x;q)\right],$$

and the corresponding mth-order deformation equation becomes

$$\mathcal{L}[u_m(x) - \chi_m \, u_{m-1}(x)] = \sum_{n=1}^{\min\{m,\kappa\}} c_n \, \delta_{m-n}(x),$$

where $\delta_k(x)$ is defined by (5). Note that the mth-order homotopy-approximation

$$u(x) \sim u_0(x) + \sum_{n=1}^{m} u_n(x)$$

contains at most the κ unknown convergence-control parameters

$$c_1, c_2, c_3, \cdots, c_\kappa.$$

Therefore, in theory, there exist a *finite* number of unknown convergence-control parameters

$$c_1, c_2, c_3, \cdots, c_\kappa$$

even as $m \to +\infty$. In this case, the optimal mth-order homotopy-approximation is given by a set of $\min\{m, \kappa\}$ nonlinear algebraic equations

$$\frac{\partial E_m}{\partial c_n} = 0, \qquad 1 \leq n \leq \min\{m, \kappa\}. \tag{14}$$

The above optimal HAM becomes exactly the so-called “optimal homotopy asymptotic method” suggested by Marinca and Herişanu [32], if $\kappa \to \infty$. Besides, when $c_1 = c_0$ and $c_n = 0$ for $n > 1$, it becomes the basic optimal HAM. Therefore, this optimal HAM is more general. For details, please refer to the book [29].