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On the numerical simulation of propagation of micro-level inherent uncertainty for chaotic dynamic systems

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ABSTRACT

In this paper, an extremely accurate numerical algorithm, namely the "clean numerical simulation" (CNS), is proposed to accurately simulate the propagation of micro-level inherent physical uncertainty of chaotic dynamic systems. The chaotic Hamiltonian Hénon–Heiles system for motion of stars orbiting in a plane about the galactic center is used as an example to show its basic ideas and validity. Based on Taylor expansion at rather high-order and MP (multiple precision) data in very high accuracy, the CNS approach can provide reliable trajectories of the chaotic system in a finite interval $t \in [0, T_c]$, together with an explicit estimation of the critical time T_c. Besides, the residual and round-off errors are verified and estimated carefully by means of different time-step Δt, different precision of data, and different order M of Taylor expansion. In this way, the numerical noises of the CNS can be reduced to a required level, i.e. the CNS is a rigorous algorithm. It is illustrated that, for the considered problem, the truncation and round-off errors of the CNS can be reduced even to the level of 10^{-1244} and 10^{-1000}, respectively, so that the micro-level inherent physical uncertainty of the initial condition (in the level of 10^{-60}) of the Hénon–Heiles system can be investigated accurately. It is found that, due to the sensitive dependence on initial condition (SDIC) of chaos, the micro-level inherent physical uncertainty of the position and velocity of a star transfers into the macroscopic randomness of motion. Thus, chaos might be a bridge from the micro-level inherent physical uncertainty to the macroscopic randomness in nature. This might provide us a new explanation to the SDIC of chaos from the physical viewpoint.

1. Introduction

Using high performance digit computers, a lot of complicated problems in science, finance and engineering have been solved with satisfied accuracy. However, there exist some problems which are still rather difficult to solve even by means of the most advanced computers. One of them is the propagation of micro-level inherent *physical* uncertainty of chaotic dynamical systems.

It is well-known that all numerical simulations are not "clean": there exist more or less numerical noises such as truncation and round-off errors, which greatly depend on numerical algorithms. In most cases, such kind of numerical noises are much larger than the micro-level inherent physical uncertainty of dynamic systems under consideration, so that the micro-level inherent uncertainty is completely lost in the numerical noise. This becomes more serious for chaotic dynamic systems, which have the sensitive dependence on initial conditions (SDIC), i.e. very tiny change of initial condition leads to great difference of numerical simulations of chaotic systems so that long-term prediction is impossible. Thus, very fine numerical algorithms need be developed to accurately simulate the

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propagation of micro-level inherent physical uncertainty of chaotic dynamic systems. This is the motivation of this article.

In this article, a kind of numerical algorithm in a rather high accuracy, called the "clean numerical simulation" (CNS), is proposed to accurately simulate propagation of micro-level inherent physical uncertainty of chaotic dynamic systems. Here, the word "clean" means that the truncation and round-off errors can be controlled to an arbitrary level that is much less than the micro-level inherent physical uncertainty of the initial condition so that the numerical noises can be neglected in a given finite interval of time for the propagation of uncertainty. A chaotic Hamiltonian system proposed by Hénon and Heiles [10] is used to show its validity. The basic ideas of the so-called clean numerical simulation (CNS) are given in Section 2, followed by the investigation of the micro-level uncertainty of the system in Section 3 and its propagation in Section 4 from statistical viewpoint. Conclusions and discussions are given in Section 5.

2. The numerical algorithm of the CNS

2.1. Basic ideas

Hénon and Heiles [10] proposed a Hamiltonian system of equations

$$\ddot{x}(t) = -x(t) - 2x(t)y(t), \tag{1}$$

$$\ddot{y}(t) = -y(t) - x^2(t) + y^2(t), \tag{2}$$

to approximate the motion of stars orbiting in a plane about the galactic center, where the dot denotes the differentiation with respect to the time t. Its solution is chaotic for some initial conditions, such as

$$x(0) = \frac{14}{25}, \quad y(0) = 0, \quad \dot{x}(0) = 0, \quad \dot{y}(0) = 0, \tag{3}$$

as mentioned by Sprott [23]. Without loss of generality, let us use this chaotic system to describe the basic ideas of the CNS and to illustrate its validity.

It is well-known [5,8,11,13,14,22,23,29] that chaotic dynamic systems have the sensitive dependence on initial conditions (SDIC), i.e. a tiny change of initial conditions leads to great difference of numerical simulations at large time, so that long-term prediction of chaos is impossible. It is well-known that all numerical simulations contain the unavoidable truncation and round-off errors at each time-step. Generally speaking, most of traditional numerical simulations of chaos are mixed with these numerical noises and thus are not "clean". Because these numerical noises of traditional numerical approaches are generally much larger than the micro-level inherent physical uncertainty of initial condition, the propagation of such kind of physical uncertainty of chaotic dynamic systems has never been studied accurately, to the best of the author's knowledge.

For numerical simulations of chaotic dynamic system, we must take rigorous account of numerical errors and rounding, because "what is observed on the computer screen would be completely unrelated to what was meant to be simulated", as pointed out by Galatolo et al. [7]. The methods of shadowing may gain accurate numerical simulations closed to true trajectories of hyperbolic dynamic systems, but fail to have long shadowing trajectories for those with a fluctuating number of positive finite-time Lyapunov exponents, as pointed out by Dawson et al. [3]. Besides, it is found that numerical simulations of chaotic systems given by low-order Runge–Kutta methods or Taylor expansion approaches have sensitive dependence not only on initial conditions but also on numerical algorithms, so that different numerical schemes might lead to completely different long-term predictions, as pointed out by Lorenz [15,16] and Teixeira et al. [20,24].

In order to gain reliable chaotic solutions in a long interval of time, Liao [12] developed a numerical technique with extremely high accuracy, called here the "clean numerical simulation" (CNS). Using the computer algebra system Mathematica with the 400th-order Taylor expansion for continuous functions and data in accuracy of 800-digit precision, Liao [12] gained, for the first time, the reliable numerical results of chaotic solution of Lorenz equation in a long interval $0 \leqslant t \leqslant 1175$ LTU (Lorenz time unit). The basic ideas of the CNS are simple and straightforward. Since the order of Taylor expansion is very high, the corresponding truncation error is rather small. Besides, since all data are expressed in the accuracy of large-number digit precision, the small enough round-off error is guaranteed. Thus, as long as the order of Taylor expansion is high enough and the digit-number of data is long enough, both of the truncation and round-off errors can be much smaller than the micro-level inherent physical uncertainty so that the propagation of micro-level uncertainty of the initial condition can be simulated accurately in a long enough interval of time. Here, the "clean" numerical simulation means that the truncation and round-off errors can be controlled to an arbitrary level so that the numerical noises can be neglected in a given *finite* interval of time, as shown later. Currently, Liao's "clean" chaotic solution [12] of Lorenz equation is confirmed by Wang et al. [27] to be a reliable trajectory of Lorenz equation in the interval $0 \leqslant t \leqslant 1175$ LTU, who used parallel computation with the multiple precision (MP) library: they gained reliable chaotic solution of Lorenz equation up to 2500 LTU by means of the 1000th-order Taylor expansion and data in the accuracy of 2100-digit precision. Note that, similar to the so-called shadowing trajectories given by the shadowing approach [21], such kind of "clean" numerical simulations given by the CNS are close to true trajectories of chaotic systems.

The CNS is based on Taylor expansion at a rather high-order. Let (x_n, y_n) and $(\dot{x}_n, \dot{y}_n)$ denote the position and velocity at the time $t_n = n\Delta t$, where Δt is a constant time-step. Assume that $x(t)$, $y(t)$ are $M+1$ times differentiable on the open interval $(t, t+\Delta t)$ and continuous on the closed interval $[t, t+\Delta t]$. According to Taylor theorem, we have

$$x(t+\Delta t) = x(t) + \sum_{n=1}^{M} a_n(t)(\Delta t)^n + R_M^x(t), \tag{4}$$

$$y(t+\Delta t) = y(t) + \sum_{n=1}^{+\infty} b_n(t)(\Delta t)^n + R_M^y(t), \tag{5}$$

where

$$a_n(t) = \frac{1}{n!}\frac{d^n x(t)}{dt^n} = \frac{x^{(n)}(t)}{n!}, \quad b_n(t) = \frac{1}{n!}\frac{d^n y(t)}{dt^n} = \frac{y^{(n)}(t)}{n!} \tag{6}$$

and

$$R_M^x(t) = a_{M+1}(\xi_1)(\Delta t)^{M+1}, \quad t \leqslant \xi_1 \leqslant t + \Delta t, \tag{7}$$

$$R_M^y(t) = b_{M+1}(\xi_2)(\Delta t)^{M+1}, \quad t \leqslant \xi_2 \leqslant t + \Delta t, \tag{8}$$

are remainders of $x(t)$ and $y(t)$, respectively. Assuming that

$$|a_{M+1}(t)| < \mu, \quad |b_{M+1}(t)| < \mu, \quad t > 0, \tag{9}$$

it holds obviously

$$\left|R_M^x(t)\right| < \mu(\Delta t)^{M+1}, \quad \left|R_M^y(t)\right| < \mu(\Delta t)^{M+1}. \tag{10}$$

Thus, we have the following theorem

Theorem of truncation error If $x(t), y(t)$ are $M + 1$ times differentiable on the open interval $(t, t + \Delta t)$ and continuous on the closed interval $[t, t + \Delta t]$, and besides if $|x^{(M+1)}(t)|/(M + 1)! < \mu$ and $|y^{(M+1)}(t)|/(M + 1)! < \mu$ for $t > 0$, where $\mu > 0$ is a constant, then the Taylor expansion

$$x(t + \Delta t) \approx x(t) + \sum_{n=1}^{M} a_n(\Delta t)^n, \tag{11}$$

$$y(t + \Delta t) \approx y(t) + \sum_{n=1}^{+\infty} b_n(\Delta t)^n, \tag{12}$$

have the truncation errors less than $\mu(\Delta t)^{M+1}$.

The round-off error is determined by the accuracy of data. To avoid large round-off error, all data are expressed in high accuracy of long-digit precision. For example, one can use data in accuracy of 2M-digit precision, where M is the order of Taylor expansions (11) and (12). Thus, for large enough M, the round-off error are rather small. For example, in case of $M = 70$, all data are expressed in accuracy of 140-digit precision so that the corresponding round-off error is in the level of 10^{-140}. Such kind of high precision data can be gained easily by means of computer algebra system like Mathematica and Maple, or the multiple precision (MP) library for FORTRAN and C. Obviously, the larger the value of M, the smaller the truncation and the round-off errors. In this meaning, we can control the truncation and round-off errors to a required level.

The coefficients a_n and b_n can be calculated in a recursive way. Assume that $a_0 = x_n, b_0 = y_n, a_1 = \dot{x}_n, b_1 = \dot{y}_n$ are known. Substituting the Taylor expansions (11) and (12) into the original governing Eqs. (1) and (2) of the Hénon and Heiles system [10] and equaling the like power of $\Delta t = t - t_n$, we have the recursion formula

$$a_{n+2} = -\frac{a_n + 2\sum_{k=0}^{n} a_k b_{n-k}}{(n+1)(n+2)}, \tag{13}$$

$$b_{n+2} = -\frac{b_n + \sum_{k=0}^{n}(a_k a_{n-k} - b_k b_{n-k})}{(n+1)(n+2)} \tag{14}$$

for $n \geqslant 0$. Then, we have the Mth-order Taylor approximation

$$x_{n+1} \approx \sum_{k=0}^{M} a_k(\Delta t)^k, \quad y_{n+1} \approx \sum_{k=0}^{M} b_k(\Delta t)^k \tag{15}$$

and

$$\dot{x}_{n+1} \approx \sum_{k=0}^{M-1}(k+1)a_{k+1}(\Delta t)^k, \tag{16}$$

$$\dot{y}_{n+1} \approx \sum_{k=0}^{M-1}(k+1)b_{k+1}(\Delta t)^k \tag{17}$$

at the time $t_{n+1} = (n + 1)\Delta t$. Besides, all data are expressed here in the accuracy of 2M-digit precision (we use the computer algebra system Mathematica). In this way, one gains rather accurate numerical simulations of $x(t)$ and $y(t)$ step by step in a finite interval of time, with extremely small truncation and round-off errors at each time-step, as verified below.

For short time, both of the truncation and round-off errors are so small that the numerical results are often close to the true trajectory. This is the reason why most of numerical results of chaotic systems given by different approaches match well in a short time from the beginning. It is widely believed by the scientific community that such kind of numerical results of chaos in a short time is reliable. However, due to the sensitivity on initial conditions of chaotic dynamic system, the truncation and round-off errors are amplified quickly so that the numerical results depart greatly from the true trajectory after a critical time T_c. Here, T_c denotes such a maximum time that numerical results gained by means of different numerical approaches (for example, with different M and Δt of the CNS) are close to the true trajectory of chaotic solution in the interval $0 \leqslant t \leqslant T_c$. In other words, the numerical results are "clean", i.e. without observable influence by the round-off and truncation errors, and thus is reliable in the finite interval $t \in [0, T_c]$. Here, the so-called critical predictable time T_c is similar to the so-called shadowing time for the shadowing approach [3,21]. Mathematically, let $u_1(t)$ and $u_2(t)$ denote two time-series given by different numerical approaches. The so-called "critical time" T_c is determined by the criteria of decoupling

$$\left|1 - \frac{u_1}{u_2}\right| > \delta, \quad \dot{u}_1\dot{u}_2 < -\epsilon, \text{ at } t = T_c, \tag{18}$$

where $\epsilon > 0$ and $\delta > 0$ are two small constants ($\epsilon = 1$ and $\delta = 5\%$ are used in this article). In this paper, the critical time T_c is determined by the CNS approach, i.e. the values of M, Δt and the accuracy of data. Obviously, the larger M, the smaller Δt and the higher accuracy of data, the longer time interval $[0, T_c]$ in which the numerical results match well with the true trajectory. For given reasonable Δt and high accuracy of data, the larger the value of M, the larger T_c. So, T_c for given M is determined by comparing the corresponding CNS result with that obtained by means of a larger value of M with the same initial condition, the same Δt and the same accuracy of data.

The key step of the CNS is to provide a good estimation of the critical time T_c, which is an important characteristic length-scale of time for the CNS. Without loss of generality, we use in this article the Mth-order Taylor expansions (11) and (12) with $\Delta t = 1/10$ and the data in accuracy of (2M)-digit precision. Comparing different CNS results given by different M, we gain the different values of T_c for different M by means of the criteria (18). Then, by means of regression analysis, it is found that T_c can be approximately expressed by

$$T_c \approx 32(1+M). \tag{19}$$

For details of how to gain the above estimation of T_c, please refer to Liao [12]. Seriously speaking, given two time series $u_1(t)$ and $u_2(t)$, different small values of ϵ andδ might give a little different value of T_c. However, it is found that the estimation expression of T_c is not sensitive to the values of ϵ and δ, mainly because chaotic systems are sensitive to numerical noises. Thus, (19) provides us a good estimation of the critical time T_c. For the sake of guarantee, it is better to choose a little larger value of M than that estimated by (19) in practice. For example, in order to gain reliable chaotic solution of the Hénon and Heiles system [10] in the interval $0 \leqslant t \leqslant 2000$, say, $T_c = 2000$, we use the 70th-order[1] Taylor expansion (with $\Delta t = 1/10$) and the data in accuracy of 140-digit precision. It should be mentioned here that (19) is consistent with the conclusion about methods of shadowing [21]: the shadowing time have power law dependencies on the level of numerical noise.

Thus, given an arbitrary value of T_c, we can always calculate such a corresponding order M of Taylor expansions that the corresponding CNS result is reliable in the interval $t \in [0, T_c]$, as verified below. In other words, given the critical time T_c, the choice of the time-step Δt and the order M of Taylor expansion for reliable trajectories in $t \in [0, T_c]$ is under control. In this meanings, the CNS approach can be regarded as a "rigorous" one.

2.2. Validity of numerical simulations

As mentioned before, the larger the order M of Taylor expansion and the more accurate the data, the better the corresponding CNS results of chaotic system (1) and (2). The CNS results at $t = 500$, 1000, 1500 and 2000 given by $M = 70$ in case of the initial condition (3) are listed in Table 1.

Is it a reliable trajectory of the chaotic system (1)–(3) in the interval $0 \leqslant t \leqslant 2000$? To verify its validity, we repeated computations by means of $\Delta t = 1/10$ and $M = 100$, 150, 200, 300, 500, respectively, and found that *all* of them give exactly the *same* trajectory in the interval $0 \leqslant t \leqslant 2000$, as listed in Table 1. Besides, even using a smaller time-setp $\Delta t = 1/20$ and $\Delta t = 1/100$ of the chaotic system (1)–(3), we *always* gain the exactly *same* trajectory in the finite interval $0 \leqslant t \leqslant 2000$ by means of $M = 100$, 150, 200, 300 and 500, respectively. All of these indicate that the CNS approach indeed provide us a *reliable* trajectory of the chaotic system (1) and (2) under the initial condition (3) in the *finite* interval $0 \leqslant t \leqslant 2000$.

To verify the CNS results, let us further consider the level of truncation and round-off errors. In case of $M = 70$ and $\Delta t = 1/10$, the round-off error is in the level of 10^{-140}. The corresponding truncation error of the CNS approach can be roughly estimated in the following way. According to our CNS results, the maximum values of $|a_{70}|$ and $|b_{70}|$ are 6.1×10^{-34} and 6.7×10^{-34}, respectively. Since two divergent series decouple quickly due to the sensitive dependence on numerical noises, the Taylor series should be convergent in the interval $t \in [0, T_c]$, i.e.

$$\frac{|a_{71}|\Delta t}{|a_{70}|} < 1, \quad \frac{|b_{71}|\Delta t}{|b_{70}|} < 1.$$

Thus, we have the estimation

$$|a_{71}| < |a_{70}|/\Delta t < 6.1 \times 10^{-33}, \quad |b_{71}| < |b_{70}|/\Delta t < 6.7 \times 10^{-33}.$$

Although there exist some uncertainty in the above deduction, we have many reasons to assume that[2]

$$|a_{71}| < 10^{-29}, \quad |b_{71}| < 10^{-29}, \tag{20}$$

i.e. $\mu = 10^{-29}$. Then, according to (10), the truncation errors should be less than 10^{-100}, which is rather small. Similarly, the truncation errors in case of $\Delta t = 1/10$ and $M = 100$, 150, 200, 300 and 500 are less than 10^{-145}, 10^{-219}, 10^{-294}, 10^{-444} and 10^{-744}, respectively, as shown in Table 2.

Similarly, in case of $M = 70$ and $\Delta t = 1/20$, the maximum CNS results of $|a_{70}|$ and $|b_{70}|$ are 6.1×10^{-34} and 6.9×10^{-34}, respectively, so that the two inequalities in (20) still hold, say, we have the same constant$\mu = 10^{-29}$ for (9) to be valid, although the smaller time step Δt is used. It is found that, in case of $M = 70$ with much smaller time-step $\Delta t = 1/100$, the corresponding maximum CNS values of $|a_{70}|$ and $|b_{70}|$ are 6.2×10^{-34} and 7.1×10^{-34}, which are very close to those found in case of $M = 70$ with $\Delta t = 1/10$ and $\Delta t = 1/20$, so that we still have the *same* constant $\mu = 10^{-29}$ for (9) to be valid! In fact, according to our

Table 1
Reliable numerical results of Hénon and Heiles' chaotic system (1)–(3) given by $M = 70$ and $\Delta t = 1/10$ with data in accuracy of 140-digit precision.

t	$x(t)$	$y(t)$
500	0.19861766	−0.23842431
1000	−0.04915404	−0.31971648
1100	−0.48949729	−0.04052161
1200	−0.04886847	0.77797491
1300	0.03097135	0.32401254
1500	0.03489977	0.43408169
2000	0.44371428	−0.30558921

Table 2
Estimated level of the truncation and round-off errors of the CNS results of the chaotic system (1)–(3) in case of $\Delta t = 1/10$.

M	Constant μ for (9)	Truncation error	Round-off error
70	10^{-29}	10^{-100}	10^{-140}
100	10^{-44}	10^{-145}	10^{-200}
150	10^{-68}	10^{-219}	10^{-300}
200	10^{-93}	10^{-294}	10^{-400}
300	10^{-143}	10^{-444}	10^{-600}
500	10^{-243}	10^{-744}	10^{-1000}

[1] The estimation formula (19) gives $M \approx 62$ for $T_c = 2000$. Considering that (19) is an estimation formula for the chaotic Hamiltonian Hénon-Heiles system, we choose $M = 70$ so as to ensure that the CNS results are indeed reliable trajectories in the interval $0 \leqslant t \leqslant 2000$.

[2] Here, we multiply the values at the right-hand side of the above expressions by 10^4 and replace the number 6.1 and 6.9 by 1.0 for the sake of simplicity.

numerical simulations based on the CNS approach, it is found that, for the same M but different time-step $\Delta t \leqslant 1/10$, the two inequalities in (9) indeed hold with the same constant μ, as shown in Tables 2–4. All of these verify the validation of (9) and therefore the correction of our estimation for the truncation errors.

Note that, the larger the order M of Taylor expansion and the smaller the time-step Δt, the smaller the truncation and round-off errors, as shown in Tables 2–4. Especially, in case of $M = 500$ and $\Delta t = 1/100$, the corresponding truncation error is in the level of 10^{-1244} and the round-off error is in the level of 10^{-1000}, respectively, which are much smaller than those given by $M = 70$ and $\Delta t = 1/10$, so that we have many reasons to believe that the numerical result given by $M = 500$ and $\Delta t = 1/100$ is much closer to the true trajectory of chaotic system (1) and (2) under the given initial condition (3). However, it should be emphasized that *all* of our CNS results given by $M \geqslant 70$ and $\Delta t \leqslant 1/10$ are the *same* as those listed in Table 1. In other words, the CNS provides us the chaotic results that are *independent* of not only the order M of Taylor expansion but also the time-step Δt and the data precision. This guarantees that our CNS results given by means of 70th-order Taylor expansion and data in accuracy of 140-digit precision are indeed a true, reliable trajectory of the chaotic dynamic system (1) and (2) with the initial condition (3), at least in the interval $t \in [0, 2000]$.

According to Tables 2–4, the truncation and round-off error of the CNS approach can be decreased to the level of 10^{-1244} and 10^{-1000} (by means of $\Delta t = 1/100$ and $M = 500$), respectively. Thus, theoretically speaking, the truncation and round-off error of the CNS approach can be reduced to a required level. Besides, the CNS results given by $\Delta t = 1/10$ and $M = 70$ agree well (in the accuracy of 8-digit precision) with *all* of the CNS results by the larger $M \geqslant 70$ and/or the smaller time-step $\Delta t \leqslant 1/10$. All of these indicate that the CNS results give the reliable trajectories of the chaotic system, and the CNS is a rigorous approach.

In addition, to show the sensitive dependence on initial condition, let us consider a different initial condition

$$x(0) = \frac{14}{25}, \quad y(0) = 10^{-60}, \quad \dot{x}(0) = 0, \quad \dot{y}(0) = 0 \tag{21}$$

with a rather tiny difference of $y(0)$, i.e. $y(0) = 10^{-60}$, from the previous initial condition (3). The corresponding CNS results given by $\Delta t = 1/10$, $M = 70$ and data in accuracy of 140-digit precision are listed in Table 5. To verify that it is a reliable trajectory of the chaotic system 1, 2 and 21 in the interval $0 \leqslant t \leqslant 2000$, we repeat the CNS approach by means of $\Delta t = 1/10$, 1/20 and $M = 100$, 150, 200, 300, 500, respectively, and *always* obtain the exactly *same* results in the interval $t \in [0, 2000]$ as those listed in Table 5. Thus, the CNS approach indeed provides the true trajectory of the chaotic dynamic system (1), (2) and (21) in the restricted interval $0 \leqslant t \leqslant 2000$. Note that, the initial condition (21) with $y(0) = 10^{-60}$ has a very tiny difference from (3) with $y(0) = 0$. According to Tables 1 and 5, the two reliable (or shadowing) trajectories corresponding respectively to the different initial conditions (3) and (21), match well each other in the interval $0 \leqslant t \leqslant 1100$. Even at $t = 1200$, they still match in accuracy of 5-digit precision. However, due to the sensitive dependance on initial condition, the two reliable (or shadowing) trajectories completely depart from each other thereafter, although their initial conditions have only a tiny difference in the micro-level 10^{-60}.

All of these indicate that the CNS results given by $\Delta t = 1/10$, $M = 70$ and data in the accuracy of 140-digit precision are indeed reliable in the interval $t \in [0, 2000]$. In other words, the CNS results given by $M = 70$ and $\Delta t = 1/10$ can be regarded as a kind of "shadowing trajectory" of the chaotic system, as mentioned by Dawson et al. [3], but in a restricted interval $0 \leqslant t \leqslant 2000$.

It should be emphasized that the difference 10^{-60} is indeed rather small, which is however much larger than the truncation error in the level of 10^{-100} and the round-off error in the level of 10^{-140} of the CNS approach. Due to this reason, the CNS provides us a tool to accurately investigate the propagation of the micro-level inherent physical uncertainty of chaotic Hénon–Heiles system, which is at the level of 10^{-60} that is much larger than the numerical noises of the CNS, as shown below.

Table 3
Estimated level of the truncation and round-off errors of the CNS results of the chaotic system (1)–(3) in case of $\Delta t = 1/20$.

M	Constant μ for (9)	Truncation error	Round-off error
70	10^{-29}	10^{-122}	10^{-140}
100	10^{-44}	10^{-176}	10^{-200}
150	10^{-68}	10^{-265}	10^{-300}
200	10^{-93}	10^{-355}	10^{-400}
300	10^{-143}	10^{-535}	10^{-600}
500	10^{-243}	10^{-895}	10^{-1000}

Table 4
Estimated level of the truncation and round-off errors of the CNS results of the chaotic system (1)–(3) in case of $\Delta t = 1/100$.

M	Constant μ for (9)	Truncation error	Round-off error
70	10^{-29}	10^{-170}	10^{-140}
100	10^{-44}	10^{-245}	10^{-200}
150	10^{-68}	10^{-369}	10^{-300}
200	10^{-93}	10^{-494}	10^{-400}
300	10^{-143}	10^{-744}	10^{-600}
500	10^{-243}	10^{-1244}	10^{-1000}

Table 5
Reliable numerical results of Hénon and Heiles' chaotic system (1) and (2) under a different initial condition (21) given by $M = 70$ and $\Delta t = 1/10$ with data in accuracy of 140-digit precision.

t	$x(t)$	$y(t)$
500	0.19861766	−0.23842431
1000	−0.04915404	−0.31971648
1100	−0.48949729	−0.04052161
1200	−0.04886067	0.77797896
1300	0.42344110	−0.24151441
1500	−0.17190612	−0.21349514
2000	0.03364286	0.17136302

3. The micro-level physical uncertainty

Many, although not all, mathematical models have clear physical background. A good model for physical problems often remains the fundamental properties and provides us a way to investigate and predict some of related physical phenomenon. For example, the law of Newtonian gravitation can describe and predict the motion of the moon or a satellite accurately. Besides, many CFD (computational fluid dynamics) software based on mathematical models can predict the flows about a ship and an airplane in an acceptable accuracy. So, many of mathematical models reveal physical truths of the related phenomenon.

Eqs. (1) and (2) provide us a model for the motion of a star orbiting in a plane about the galactic center, which has very clear physical background. In general, a good mathematical model should remain the key physical characteristics of the corresponding natural phenomena. Since the Hénon–Heiles system has been widely accepted by scientific community, we have many reasons to believe that (1) and (2) as a mathematical model process the fundamental physical characteristics of the motion of a star orbiting in a plane about the galactic center.

The kinetic status of a star is determined by its position and velocity. In the frame of Newtonian gravity law, it is believed that the kinetic status of a star is *inherently* exact and the uncertainty of position and velocity come from the imperfect measure equipments which provide limited knowledge. However, according to de Broglie [4], this traditional idea is wrong: the position of a star contains *inherent* uncertainty. Besides, the quantum fluctuation might influence the existence of the so-called "objective randomness", which is independent of any experimental accuracy of the observations or limited knowledge of initial conditions, as suggested by Consoli et al. [2]. Furthermore, "all the sources of complexity examined so far are actually channels for the amplification of naturally occurring randomness in the physical world", as suggested by Allegrini et al. [1].

It is a common belief of the scientific community that the microscopic phenomenon are essentially uncertain and random. To show this point, let us consider some typical length scales of microscopic phenomenon widely used in modern physics. For example, Bohr radius

$$r = \frac{\hbar^2}{m_e e^2} \approx 5.2917720859(36) \times 10^{-11}\ (\mathrm{m})$$

is the approximate size of a hydrogen atom, where ħ is a reduced Planck's constant, m_e is the electron mass, and e is the elementary charge, respectively. Besides, Compton wavelength $L_c = \hbar/(mc)$ is a quantum mechanical property of a particle, i.e. the wavelength of a photon whose energy is the same as the rest-mass energy of the particle, where m is the rest-mass of the particle and c is the speed of light. It is the length scale at which quantum field theory becomes important. The value for the Compton wavelength of the electron is

$$L_c \approx 2.4263102175(33) \times 10^{-12}\ (\mathrm{m}).$$

In addition, the Planck length

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616252(81) \times 10^{-35}\ (\mathrm{m}) \tag{22}$$

is the length scale at which quantum mechanics, gravity and relativity [6] all interact very strongly, where c is the speed of light in a vacuum, G is the gravitational constant, and ħ is the reduced Planck constant. Especially, according to the string theory [19], the Planck length is the order of magnitude of the oscillating strings that form elementary particles, and *shorter length do not make physical senses*. Besides, in some forms of quantum gravity, it becomes *impossible* to *determine* the difference between two locations less than one Planck length apart. Therefore, in the accuracy of the Planck length level, the position of a star is inherently uncertain, so is its velocity. Note that this kind of microscopic physical uncertainty is *inherent* and has nothing to do with the Heisenberg uncertainty principle [9] and the ability of human being.

On the other hand, according to de Broglie [4], any a body has the so-called wave-particle duality, and the length of the so-called de Broglie wave is given by

$$\lambda = \frac{h}{mv}\sqrt{1 - \left(\frac{v}{c}\right)^2}, \tag{23}$$

where m is the rest mass, v denotes the velocity of the body, c is the speed of light, h is the Planck's constant, respectively. Note that, the de Broglie's wave of a body has non-zero amplitude, meaning that the position is uncertain: it could be almost *anywhere* along the wave packet. Thus, according to the de Broglie's wave-particle duality, the position of a star is *inherent* uncertain, too.

Therefore, it is reasonable for us to assume that the micro-level inherent fluctuation of position of a star shorter than the Planck length l_p is essentially uncertain and/or random.

To gain the dimensionless Planck length l_p, we use the dimeter of Milky Way Galaxy as the characteristic length, say, $d_M \approx 10^5$ (light year) $\approx 9 \times 10^{20}$ (m). Obviously, $l_p/d_M \approx 1.8 \times 10^{-56}$ is a rather small dimensionless number. As mentioned above, two (dimensionless) positions shorter than 10^{-56} do not make physical senses. Thus, it is reasonable to assume the existence of the inherent uncertainty of the dimensionless position and velocity of a star in the normal distribution with zero mean and the micro-level standard deviation 10^{-60}. Strictly speaking, such kind of micro-level inherent physical uncertainty should be added to the *observed* values (x_0, y_0, u_0, v_0) of the initial conditions, especially for chaotic dynamic systems whose solutions are rather sensitive to initial conditions.

Therefore, strictly speaking, the initial condition should be expressed as follows

$$x(0) = x_0 + \tilde{x}_0, \quad y(0) = y_0 + \tilde{y}_0, \quad \dot{x}(0) = u_0 + \tilde{u}_0, \quad \dot{y}(0) = v_0 + \tilde{v}_0,$$

where x_0, y_0, u_0, v_0 are observed values of the initial position and velocity of a star orbiting in a plane about the galactic center, and $\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0$ are the corresponding *micro-level* inherent uncertain ones, respectively. Assume that (x_0, y_0, u_0, v_0) is exactly given and the inherent uncertain term $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ is in the normal distribution with zero mean and a micro-level deviation $\sigma = 10^{-60}$. The reasons for the above assumptions are described above.

Compared to the scale of the initial data $x_0 = 14/25$, the deviation 10^{-60} is indeed rather small. However, by means of the CNS approach with the 70th-order Taylor expansion and the MP data in accuracy of 140-digit precision, the propagation of such kind of micro-level uncertainty can be accurately studied, because the corresponding truncation error (in the level of 10^{-100}) and round-off error (in the level of 10^{-140}) of the CNS approach is much smaller than the micro-level uncertainty (in the level of 10^{-60}), as verified in Section 2.

4. Statistic property of chaos

Without loss of generality, let us consider the case of the *observed* values

$$x_0 = \frac{14}{25}, \quad y_0 = 0, \quad u_0 = 0, \quad v_0 = 0$$

of the initial conditions, corresponding to a chaotic motion [22]. The so-called *observed* values can be regarded as the mean of measured data. Let

$$\langle x(t)\rangle = \frac{1}{N}\sum_{i=1}^{N} x_i(t), \quad \sigma_x(t) = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}[x_i(t) - \langle x(t)\rangle]^2}$$

denote the sample mean and unbiased estimate of standard deviation of $x(t)$, respectively, where $N = 10^4$ is the number of total samples, $x_i(t)$ is the ith sample given by the CNS using $\Delta t = 1/10$, $M = 70$ with the MP data in accuracy of 140-digit precision, and a tiny random term $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ with the micro-level deviation $\sigma = 10^{-60}$ in the initial condition.

According to Section 2, all of these 10^4 trajectories given by the CNS approach are reliable in the interval $t \in [0, 2000]$. The standard deviations $\sigma_x(t)$ and $\sigma_y(t)$ of $x(t)$, $y(t)$ are as shown in Figs. 1 and 2, respectively. Note that there exists an interval $0 \leqslant t \leqslant T_d$ with $T_d \approx 1000$, in which $\sigma_x(t)$ and $\sigma_y(t)$ are in the level of 10^{-14} so that one can accurately predict the position (x,y) of a star, even if the corresponding motion is chaotic and the initial condition contains uncertainty. Similarly, the velocity of the star can be also precisely predicted in $0 \leqslant t \leqslant T_d$, as shown in Fig. 3 for the standard deviation $\sigma_u(t)$ of $\dot{x}(t)$. Thus, when $0 \leqslant t \leqslant T_d$, the behavior of the chaotic system looks like "deterministic" and "predictable", even from the statistic viewpoint. When $t > T_d$, the standard deviations of the position and velocity begin to increase rapidly, and thus the system becomes random obviously: the position (x,y) and velocity $(\dot{x}, \dot{y})$ of the star are strongly dependent upon their

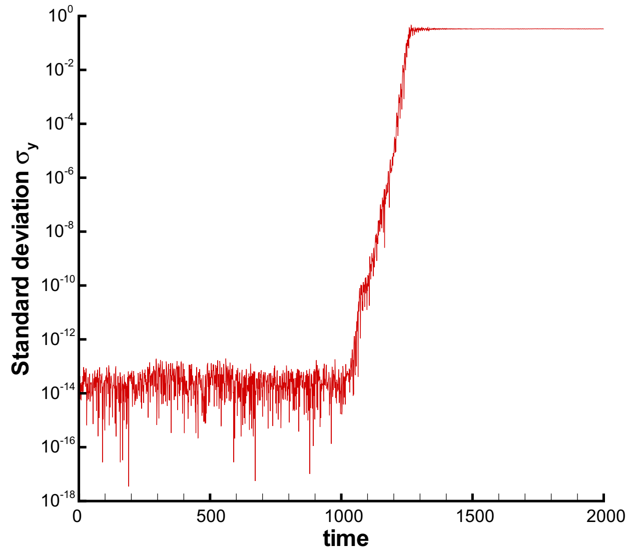

Fig. 2. The standard deviation σ_y of y in case of $x_0 = 14/25$, $y_0 = 0$, $u_0 = 0$, $v_0 = 0$ and the uncertain term $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ in the normal distribution with zero mean and a micro-level deviation $\sigma = 10^{-60}$.

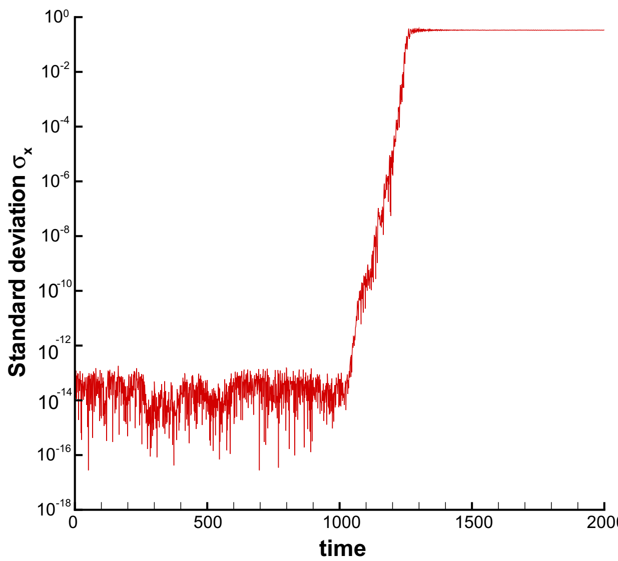

Fig. 1. The standard deviation σ_x of x in case of $x_0 = 14/25$, $y_0 = 0$, $u_0 = 0$, $v_0 = 0$ and the uncertain term $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ in the normal distribution with zero mean and a micro-level deviation $\sigma = 10^{-60}$.

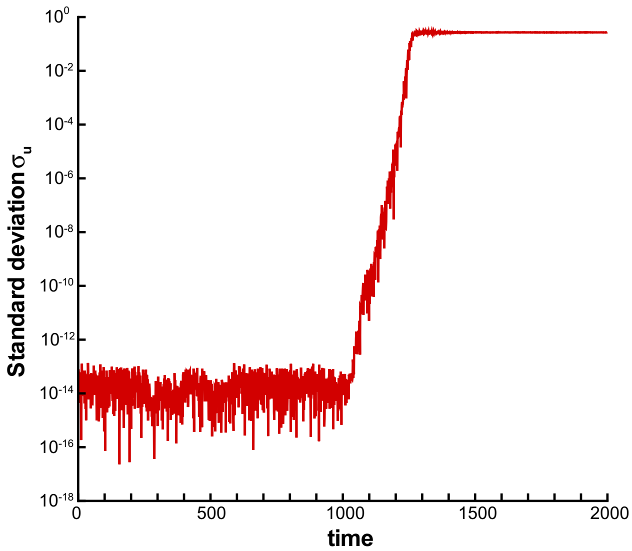

Fig. 3. The standard deviation σ_u of $\dot{x}$ in case of $x_0 = 14/25$, $y_0 = 0$, $u_0 = 0$, $v_0 = 0$ and the uncertain term $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ in the normal distribution with zero mean and a micro-level deviation $\sigma = 10^{-60}$.

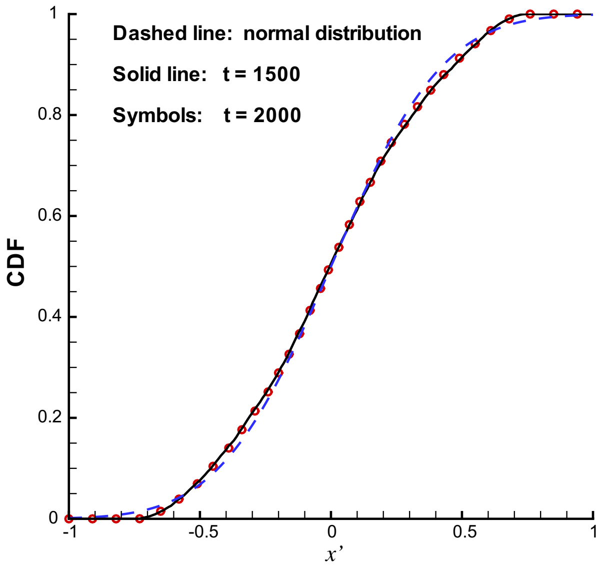

Fig. 4. The CDF of x', compared to the normal distribution (dashed line) with zero mean and the standard deviation of x' at $t = 2000$. Solid line: CDF of x' at $t = 1500$; symbols: CDF of x' at $t = 2000$.

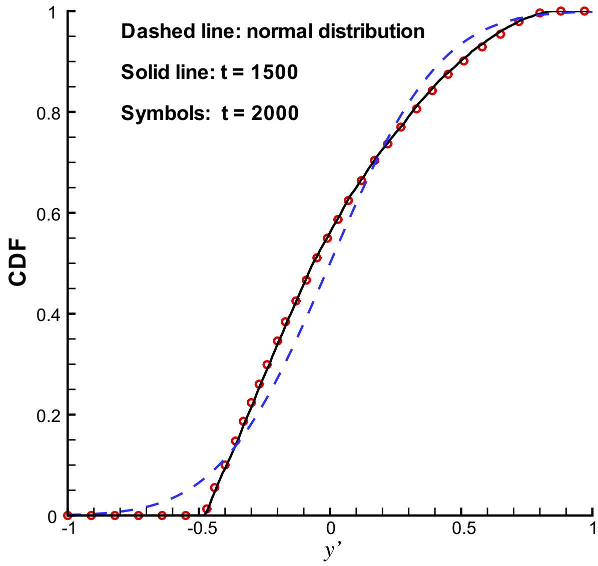

Fig. 5. The CDF of y', compared to the normal distribution (dashed line) with zero mean and the standard deviation of y' at $t = 2000$. Solid line: CDF of y' at $t = 1500$; symbols: CDF of y' at $t = 2000$.

micro-level inherent physical uncertainty $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ of the initial condition. In other words, due to the SDIC of chaos, the unobservable micro-level inherent uncertainty of the position and velocity of a star transfers into the macroscopic randomness of the motion. This suggests that chaos might be a bridge from the micro-level uncertainty to macroscopic randomness! Therefore, the micro-level inherent uncertainty of the position and velocity might be an origin of the macroscopic randomness of motion of stars in our universe. Possibly, this might provide us a new, *physical* explanation and understanding for the SDIC of chaos. For this reason, each "big bang" [18] will create a completely different universe!

Besides, it is found that the standard deviations of the position and velocity become almost stationary when $t > T_s$, where $T_s \approx 1300$, as shown in Figs. 1–3. Thus, when $T_d < t < T_s$, the system is in the transition process from the "deterministic" behavior to the stationary randomness. It is interesting that the stationary standard deviations of $x(t)$ and $y(t)$ are about 1/3, and their stationary means $\langle x \rangle$ and $\langle y \rangle$ are close to zero. It means that, due to SDIC of chaos and the micro-level inherent uncertainty of position and velocity, a star orbiting in a plane about the galactic center could be almost *everywhere* in the galaxy at a given time $t > T_s$.

Write the fluctuations $x'(t) = x - \langle x \rangle$ and $y'(t) = y - \langle y \rangle$. The stationary cumulative distribution functions (CDF) of x', y' are almost independent of time, as shown in Figs. 4 and 5. Besides, the stationary CDF of the fluctuation x' is rather close to the normal distribution with zero mean and the standard deviation of x', as shown in Fig. 4. But, the stationary CDF of the fluctuation y' is obviously different from the normal distribution, as shown in Fig. 5.

Similarly, we investigate the influence of the observed values (x_0, y_0, u_0, v_0) and the standard deviation σ of the uncertain terms $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ in the initial condition by means of the CNS approach. It is found that T_d decreases exponentially with respect to σ. Besides, the stationary means and standard deviations of $x, y, \dot{x}, \dot{y}$, and the CDFs of x' and y', are independent of the observed values (x_0, y_0, u_0, v_0). Thus, when $t > T_s$, all observed information of the initial condition are lost completely. In other words, when $t > T_s$, the asymmetry of time breaks down so that the time has a one-way direction, i.e. the arrow of time. So, statistically speaking, the Hénon–Heiles system has two completely different dynamic behaviors before and after T_d: it looks like "deterministic" and "predictable" without time's arrow when $t \leqslant T_d$, but thereafter rapidly becomes obviously random with a arrow of time.

Consoli et al. [2] suggested that the objective randomness "might introduce a weak, residual form of noise which is intrinsic to natural phenomena and could be important for the emergence of complexity at higher physical levels". Our extremely accurate numerical simulations based on the CNS approach support their viewpoint: the micro-level uncertainty and the macroscopic randomness might have a rather close relationship.

5. Conclusions and discussions

In this paper, an extremely accurate numerical algorithm, namely the "clean numerical simulation" (CNS), is proposed to accurately simulate the propagation of micro-level inherent physical uncertainty of chaotic dynamic systems. The chaotic Hénon–Heiles system describing the motion of a star orbiting in a plane about the galactic center is used as an example to show the validity of the CNS approach.

In the frame of the CNS approach, the truncation error is estimated by (9), the round-off error is determined by the digit-length of data, and the critical time T_c is explicitly determined by (19) (for the chaotic Hénon–Heiles system). So, given an arbitrary value of T_c, we can always find out the required order M of Taylor expansion and the data in accuracy of $2M$-digit precision so as to gain a reliable trajectory of the chaotic Hénon-Heiles system in the finite interval $t \in [0, T_c]$ by means of $\Delta t = 1/10$. In addition, the CNS results in the interval $t \in [0,T_c]$ are verified very carefully by means of Taylor expansion at higher-order and MP data in more accuracy, as shown in Section 2. As shown in Section 2, *all* of the CNS results (for the same initial condition) given by $\Delta t = 1/10$, $M = 70$ and data in accuracy of 140-digit precision are exactly the *same* as those given by $M = 100$, 150, 200, 300, 500 and $\Delta t = 1/20$, 1/100, respectively, so that they are indeed reliable, true trajectories of the chaotic Hénon–Heiles system. Besides, as the order M of Taylor expansion increases and the time-step Δt decreases, the truncation and round-off errors decrease monotonously. For example, as illustrated in Tables 2–4, the truncation error is in the level of 10^{-100} in case of $\Delta t = 1/10$ and $M = 70$, and decreases to the level 10^{-1244} in case of $\Delta t = 1/100$ and $M = 500$. In addition, the round-off error is simply in the level of 10^{-2M}, where M denotes the order of Taylor expansion. So, theoretically speaking, one can control the truncation and round-off error to a required level. In these meanings, the CNS approach is a rigorous one.

The Hénon–Heiles system of (1) and (2) as a mathematical model has clear physical background: it has been widely accepted and used by the scientific community to describe the motion of a star orbiting in a plane about the galactic center. The status of a star is dependent upon its position and velocity. However, according to de Broglie [4], the position of a star contains micro-level *inherent* physical uncertainty, as discussed in Section 3. So, strictly speaking, the Hénon–Heiles system of (1) and (2) is *not* deterministic in essence. Due to the SDIC of chaos, such kind of micro-level physical uncertainty transfers into macroscopic randomness of motion, as illustrated in Section 4 by means of the CNS approach. Therefore, the micro-level inherent physical uncertainty and macroscopic randomness might have a close relationship: chaos might be a bridge from the micro-level inherent physical uncertainty to macroscopic randomness! This conclusion agrees with the viewpoint of Consoli et al. [2] who suggested that the objective randomness "might introduce a weak, residual form of noise which is intrinsic to natural phenomena and could be important for the emergence of complexity at higher physical levels".

The CNS approach provides us an extremely precise numerical approach for chaotic dynamic systems in a given *finite* interval $t \in [0, T_c]$. According to (19), $T_c \to +\infty$ as $M \to +\infty$. In other words, if the initial condition were exact, then long-term prediction of chaos would be possible *in theory*[3]: given an arbitrary value of T_c, we could gain the reliable chaotic trajectory of the Hénon–Heiles system in the *finite* interval $0 \leqslant t \leqslant T_c$ by means of the Mth-order Taylor expansion with data in accuracy of $2M$-dight precision, as long as $M > T_c/32$ and $\Delta t \leqslant 1/10$. Qualitatively, the conclusion has general meanings and holds for other chaotic models such as Lorenz equation. Besides, it is consistent with Tucker's elegant proof [25,26] that there indeed exists an attractor of Lorenz equation. Thus, theoretically speaking, there is no place for the randomness in a *truly* deterministic system. However, most models related to physical problems contain more or less physical uncertainty, and thus, strictly speaking, are *not* deterministic. For such kind of physical models with inherent uncertainty, the accurate long-term prediction of trajectories of chaotic system has *no* physical meanings, because their long-term trajectories are *inherently* random that comes from the micro-level inherent physical uncertainty, as illustrated in this article.

Traditionally, it is believed that the long-term prediction of chaos is impossible, mainly due to the impossibility of the perfect *measurement* of initial conditions with an *arbitrary* degree of accuracy. This is the traditional explanation to the SDIC of chaos. Here, we provide a new explanation for the SDIC of chaos from the physical viewpoint: initial conditions of some chaotic systems with clear physical meanings might contain micro-level *inherent* physical uncertainty, which might propagate into macroscopic randomness. Different from the traditional explanation of the SDIC, which focuses on the *measurement*, the new explanation emphasizes the *inherent* micro-level uncertainty and its propagation with chaos. Besides, it should be emphasized that such micro-level inherent physical uncertainty of chaos was completely inundated with the numerical noises of the traditional numerical methods based on low-order algorithms, and thus has never been studied in details. This shows the validity and potential of the CNS to precisely simulate complex physical phenomena with the SDIC, such as weather prediction and turbulence.

Finally, for the easier understanding of the CNS, let us consider the map

$$f(x) = \mathrm{mod}(2x, 1) \tag{24}$$

with the initial value $x_0 = \pi/4$. It is well-known that this map has the sensitivity dependence on initial condition, i.e. SDIC. The results of the nth mapping, i.e. $x_n = f(x_{n-1})$ with $x_0 = \pi/4$, are expressed by both of the decimal and binary systems in Table 6. In binary system, the mapping x_n corresponds to such a kind of left shift: shifting x_0 left to the position of its 2nd digit "1" gives x_1, and to the position of its 3rd digit "1" gives x_2, and so on, as shown in Table 6. In general, x_n (in binary system) corresponds to the left shift of x_0 (in binary system) to its position of the $(n+1)$ th digit "1". Since $\pi/4$ is exactly known in binary system, its position of the nth digit "1" is deterministic, denoted by $P_2(n)$. So, in binary system, x_n is exactly the left shift of x_0 to its $P_2(n+1)$th digit "1". So, mathematically

Table 6
Mapping of $f(x) = \mathrm{mod}\,(2x,1)$ with $x_0 = \pi/4$, expressed in decimal and binary systems.

n	x_n in decimal system	x_n in binary system
0	0.785398163397448...	11001001000011111101101010l...
1	0.570796326794896 ...	100100100001111110110 10 ...
2	0.141592653589793 ...	10010000111111011010 ...
3	0.283185307179586 ...	10000111111011010 ...
⋮	⋮	⋮

[3] Unfortunately, the required CPU time increases exponentially as M increases, so that it is *practically* impossible to give reliable, true trajectories of chaos in a very large interval.

speaking, this mapping is deterministic and x_n is exactly known. However, in practice, one had to take $x_0 = \pi/4$ in a *finite* accuracy, which leads to uncertainty. Assume that x_0 is in accuracy of N_0 binary digits. Then, x_1, x_2, x_3 has $N_0 - 1$, $N_0 - 4$, $N_0 - 7$ significance binary digits, respectively, as shown in Table 6. In general, x_n is in the accuracy of $N_0 - P_2(n+1)$ binary digit precision. Obviously, when $P_2(n+1) > N_0$, the mapping x_n losses its accuracy at all. However, even at one million times of mapping, i.e. $n = 10^6$, we can gain the *accurate enough* result $x_{1000000}$ in the accuracy of one million of binary digit precision, as long as we take the initial value $x_0 = \pi/4$ in the accuracy of $10^6 + P_2(10^6+1)$ binary digit precision! This simple example illustrates that we *do* can gain reliable results for dynamic systems with SDIC in a *finite* times of mapping or a *finite* interval, as long as initial conditions are *accurate enough*. This also explains why the CNS is based on rather accurate data, using the computer algebra system Mathematica or the multiple precision library.

However, a chaotic dynamic system has no such kind of elegant property of mod $(2x, 1)$ mentioned above, since its exact solution is unknown in general. Thus, the above approach based on the left shift has no general meanings. Assume that one knows the SDIC of the mapping $f(x) = \text{mod}\,(2x, 1)$, but has no ideas about its left-shift property in the corresponding binary system. How to gain reliable sequence $x_n = f(x_{n-1})$ by means of $x_0 = \pi/4$? A general, straight-forward way is to compare two sequences given by $x_0 = \pi/4$ in different accuracy of N-digit precision (in decimal system), where $N = 15$, 20, 25, 30 and 1000, respectively, as shown in Table 7. For example, by comparing the two sequences of x_n given by x_0 in accuracy of 15 and 20-digit precision, one is sure due to the SDIC that the sequence x_n given by x_0 in accuracy of 15-digit precision is reliable at $n \leqslant 15$ in accuracy of 8 significance digits. Similarly, using x_0 in accuracy of 20 and 25-digit precisions, one gains reliable x_n at $n \leqslant 30$ and $n \leqslant 40$ with 8 significance digits, respectively. Note that the sequence x_n given by x_0 in accuracy of 25-digit precision agrees well with that by x_0 in accuracy of 30-digit precision for a *finite* number of mappings x_n, where $0 \leqslant n \leqslant 40$. Thus, one has many reasons to believe that the *finite* sequence x_0, x_1, ..., x_{40} given by means of x_0 in accuracy of 25-digit precision is reliable. This is indeed true, because it completely agrees with the "exact" sequence given by x_0 in accuracy of 1000-digit precision, as shown in Table 7. The key point is that, to gain reliable sequence $x_0, x_1, \ldots, x_{40}$ with the *finite* number of mappings, we need use x_0 in accuracy of *only* 25-digit precision: it is unnecessary to use x_0 in higher accuracy. Similarly, using x_0 in accuracy of 40-digit precision, one can gain reliable sequence

$$x_0, x_1, x_2, \cdots, x_{100}$$

in accuracy of 8 significance digits. Furthermore, using x_0 in accuracy of 60-digit precision, one can gain reliable sequence

$$x_0, x_1, x_2, \ldots, x_{166}$$

in accuracy of 8-digit precision as well. And so on. Thus, we can gain reliable x_n of *finite but many enough* mappings by using *accurate enough* x_0. In other words, the reliability and precision of the *finite* sequence

$$x_0, x_1, x_2, \ldots, x_n$$

given by the mapping $f(x) = \text{mod}\,(2x, 1)$ with SDIC is under *control*. It is true that, using x_0 in 60-digit precision, x_{100000} is incorrect and thus has no meaning. However, the key point is that the corresponding sequence

$$x_0, x_1, x_2, \ldots, x_{166}$$

of a *finite* number of mappings is reliable in accuracy of 8-digit precision, which might be enough for one's purpose. In essence, we seek for a kind of *relative* reliability and predictability of chaotic dynamic systems, although very long-term accurate prediction of any a chaotic dynamic system is absolutely impossible in theory. It is true that, using x_0 in accuracy of any a given precision, there *absolutely* exists such a large enough n that x_n totally losses its accuracy. However, we can guarantee the reliability and predictability of a given *finite* sequence (such as $x_0, x_1, x_2, \ldots, x_{166}$) by using x_0 in a reasonable accuracy (such as 60-digit precision). It should be emphasized that such kind of comparison approach is valid for any chaotic dynamic systems. So, it has general meanings and thus is practical. Note that the *same* comparison approach is used in the CNS described in Section 2 and Section 3. This example clearly explains why the CNS based on such kind of comparison is indeed reasonable and valid.

It is important to provide a *practical* numerical approach to gain reliable chaotic solutions of dynamic systems in a long enough interval of time. Using CNS with 400-order Taylor expansion, data in accuracy of 800-digit precisions and $\Delta t = 10^{-2}$, Liao [12] gained, for the first time, a reliable chaotic solution of Lorenz equation in a

Table 7
x_n given by the mapping $f(x) = \text{mod}\,(2x, 1)$ with the initial value $x_0 = \pi/4$ in different accuracy of N decimal digit precision.

n	$N = 15$	$N = 20$	$N = 25$	$N = 30$	$N = 1000$
5	0.1327412287	0.1327412287	0.1327412287	0.1327412287	0.1327412287
10	0.2477193189	0.2477193189	0.2477193189	0.2477193189	0.2477193189
15	0.9270182076	0.9270182075	0.9270182075	0.9270182075	0.9270182075
20	0.664582643	0.6645826427	0.6645826427	0.6645826427	0.6645826427
25	0.2666446	0.2666445682	0.2666445682	0.2666445682	0.2666445682
30	0.532626	0.5326261849	0.5326261849	0.5326261849	0.5326261849
35	0.0440	0.044037917	0.0440379171	0.0440379171	0.0440379171
40	0.409	0.40921335	0.4092133503	0.4092133503	0.4092133503

rather long time interval $0 \leqslant t \leqslant 1000$, whose correction is confirmed by Wang et al. [27]. As mentioned by Wang [28], in order to gain reliable chaotic solution of Lorenz equation in the interval $0 \leqslant t \leqslant 1000$ by means of the 4th-order Runge–Kutta method, one had to use multiple precision data in 10000-digit precision and a rather small time-step $\Delta t = 10^{-170}$, which however needs about 3.1×10^{160} years by today's high-performance computer! Therefore, the low-order Taylor expansion approaches are *not* practical to gain reliable chaotic solution of Lorenz equation in such a long time interval. There exist some "rigorous" simulations [26] assuring that the real orbits of chaotic system are "enclosed" in a computed region of space, such as $[x(t) - \delta, x(t) + \delta]$, where δ should be a small constant: results with large δ is useless in practice, even though it is obtained by "rigorous" methods. Due to SDIC, it is obvious that one had to use rather small δ to gain such a rigorous chaotic solution of Lorenz equation in $0 \leqslant t \leqslant 1000$ by means of the enclosing approach: possibly δ might be in the level of 10^{-480}, since the corresponding initial condition must be accurate in 480 digit precision, as pointed out by Liao [12]. However, to the best of author's knowledge, it is still an *open* question whether or not the "rigorous" method based on enclosing [26] can give such a reliable, accurate enough chaotic solution of Lorenz equation in the interval $0 \leqslant t \leqslant 1000$ by means of a reasonable CPU time. Besides, to the best of author's knowledge, it is also an *open* question whether or not the enclosing approach is *practical* for physical problems like those considered in this article: note that the CNS is successfully used to gain 10000 samples of reliable chaotic solutions given by different initial conditions with 10^{-60}-level uncertainty. So, compared to other approaches, the CNS is not only *reliable* but also *practical*.

Indeed, the propagation of round-off and truncation errors of a chaotic dynamic system is rather complicated and thus is unknown in general cases. As pointed out by Parker and Chua [17], a "practical" way of judging the accuracy of numerical results of a non-linear dynamic system is to use at least two (or more) "different" routines to integrate the "same" system. This is mainly because, due to the SDIC of chaotic dynamics systems, departure of two chaotic simulations indicates the appearance of large enough truncation and round-off errors. In practice, the comparison approach provides us a time interval $0 \leqslant t \leqslant T_c$, in which the same results should be reliable, mainly due to SDIC of chaos. Certainly, such kind of critical time T_c must be carefully checked by as many different approaches as possible, as shown in Section 2.2. In fact, such kind of comparison approach is widely accepted by scientific community [17,24,27,28]. And the CNS is based on such kind of strategy. Using a metaphor, it is like measuring the height of a man: the better the equipment, the more accurate the result, although we can not provide an "exact" value of the height. Although it is difficult to measure the height of a man in accuracy of 10^{-10} meter, it is rather easy to ensure that whether a man is higher than 1.85 m or not, as long as all measures given by all equipments give us the same answer to this question: such kind of precision is relatively rough but enough in many cases of everyday life. Similarly, the CNS seeks for reliable, accurate enough simulations of chaotic dynamic systems in a *finite* time-interval.

In summary, the CNS provides us a *practical* way to gain reliable, *accurate enough* solutions of chaotic dynamic systems with a *high enough* precision in a *finite* but *long enough* time interval.

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