



ELSEVIER Applied Mathematics and Computation 147 (2004) 499–513

Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE @ DIRECT®

APPLIED
MATHEMATICS
AND
COMPUTATION

www.elsevier.com/locate/amc

On the homotopy analysis method for nonlinear problems

Shijun Liao

*School of Naval Architecture and Ocean Engineering, Shanghai Jiao Tong University,
Shanghai 200030, China*

Abstract

A powerful, easy-to-use analytic tool for nonlinear problems in general, namely the homotopy analysis method, is further improved and systematically described through a typical nonlinear problem, i.e. the algebraically decaying viscous boundary layer flow due to a moving sheet. Two rules, the rule of solution expression and the rule of coefficient ergodicity, are proposed, which play important roles in the frame of the homotopy analysis method and simplify its applications in science and engineering. An explicit analytic solution is given for the first time, with recursive formulas for coefficients. This analytic solution agrees well with numerical results and can be regarded as a definition of the solution of the considered nonlinear problem.

© 2002 Elsevier Inc. All rights reserved.

Keywords: Homotopy analysis method; Analytic; Nonlinear; Similar boundary-layer; Stretching wall

1. Introduction

In most cases it is difficult to solve nonlinear problems, especially analytically. Perturbation techniques [1,2] are currently the main stream. Perturbation techniques are based on the existence of small/large parameters, the so-called perturbation quantity. Unfortunately, many nonlinear problems in science and engineering do not contain such kind of perturbation quantities at all. Some nonperturbative techniques, such as the artificial small parameter method [3],

E-mail address: sjliao@mail.sjtu.edu.cn (S. Liao).

the δ -expansion method [4] and the Adomian's decomposition method [5], have been developed. Different from perturbation techniques, these nonperturbative methods are independent upon small parameters. However, both of the perturbation techniques and the nonperturbative methods *themselves* can *not* provide us with a simple way to *adjust* or *control* the convergence region and rate of given approximate series.

Liao [6] proposed a powerful analytic method for nonlinear problems, namely the homotopy analysis method [7–13]. Different from *all* reported perturbation and nonperturbative techniques mentioned above, the homotopy analysis method *itself* provides us with a convenient way to control and adjust the convergence region and rate of approximation series, when necessary. Briefly speaking, the homotopy analysis method has the following advantages

- it is valid even if a given nonlinear problem does *not* contain any small/large parameters *at all*;
- it *itself* can provide us with a convenient way to adjust and control the convergence region and rate of approximation series when necessary;
- it can be employed to *efficiently* approximate a nonlinear problem by *choosing* different sets of base functions.

To systematically describe the basic ideas of the homotopy analysis method and to show its validity, let us consider a viscous boundary layer flow due to a moving sheet occupying the negative x -axis and moving continuously in the positive x -direction at a velocity

$$u_s = u_0 \left(\frac{x_0}{|x|} \right)^\kappa, \quad 0 < \kappa < 1, \quad (1)$$

where (x, y) denotes the coordinate in Cartesian system. The boundary layer flow is governed by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}, \quad (2)$$

where u and v are the velocity components in the x - and y -directions, respectively. The corresponding boundary conditions are

$$u = u_s, \quad v = 0 \quad \text{at } y = 0, \quad u \rightarrow 0 \quad \text{as } y \rightarrow +\infty. \quad (3)$$

Under the similar transformation

$$\psi = F(\xi) \sqrt{2\nu u_s |x|}, \quad \xi = y \sqrt{\frac{u_s}{2\nu |x|}}, \quad (4)$$

where ψ is the stream function defined by $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, the Eqs. (2) and (3) become

$$F'''(\xi) + (\kappa - 1)F(\xi)F''(\xi) - 2\kappa[F'(\xi)]^2 = 0 \quad (5)$$

and

$$F(0) = 0, \quad F'(0) = 1, \quad F'(+\infty) = 0, \quad (6)$$

where the prime denotes differentiation with respect to ξ . For details, please refer to Kuiken [14].

Kuiken [14] gave such an asymptotic expression

$$f \sim (\xi - \xi_0)^\alpha \sum_{i=0}^N c_0^{1-i} \bar{c}_i (\xi - \xi_0)^{-i(1+\alpha)}, \quad (7)$$

where

$$\alpha = \frac{1 - \kappa}{1 + \kappa} \quad (8)$$

and the coefficients \bar{c}_i are given by recursive formulas and the coefficients c_0, ξ_0 are determined by an iterative numerical approach. Thus, rigorously speaking, Kuiken's solution is semi-analytic and semi-numerical one. Besides, the above expression is valid only for $\xi - \xi_0 \gg 1$, because it is singular at $\xi = \xi_0$. To the best of our knowledge, no one has reported an explicit, *purely* analytic solution of (5) and (6), valid in the *whole* region $0 \leq \xi \leq +\infty$.

In this paper the homotopy analysis method is further improved and systematically described in a usual procedure through a typical example mentioned above. Two rules are described, which play important roles in the frame of the homotopy analysis method and simplify its applications in science and engineering. An explicit analytic solution of above nonlinear problem is given for the first time.

2. Homotopy analysis method

In this section the homotopy analysis method is further improved and systematically described to give an explicit analytic solution of the nonlinear problem mentioned above. A usual procedure of the homotopy analysis method is proposed for the first time.

2.1. Analysis of asymptotic property

The application of the homotopy analysis method starts from the analysis of asymptotic property of the considered problem, if possible. Due to the boundary condition (6), $F'(\xi) \rightarrow 0$ as $\xi \rightarrow +\infty$. So, it is important to qualitatively analyze the asymptotic property of $F(\xi)$ at infinity. Does $F'(\xi) \rightarrow 0$

exponentially or algebraically? Kuiken [14] pointed out that when $0 < \kappa < 1$ Eqs. (5) and (6) have solutions with the algebraic property

$$F(\xi) \sim \xi^\alpha \quad (9)$$

for large ξ , where α defined by (8) is obtained by substituting the main term $F \sim \xi^\alpha$ into (5) for large ξ . Thus, $F'(\xi) \sim \xi^{\alpha-1}$ algebraically decays to zero as $\xi \rightarrow +\infty$.

The analysis of asymptotic property of a nonlinear problem often provides us with a lot of valuable information, which often greatly increase convergence rate of approximate series. However it had to be pointed out that sometimes it is hard to analyze asymptotic properties of a given nonlinear problem at infinity.

2.2. Rule of solution expression

Due to the asymptotic property (9) and Eqs. (5) and (6), it is natural to assume that $F(\xi)$ can be expressed by the set of base function

$$\{(1 + \xi)^\alpha, 1, (1 + \xi)^{m\alpha - n} \mid m\alpha - n < 0, m \geq 1 \text{ and } n \geq 1 \text{ are integers}\} \quad (10)$$

so that $F(\xi)$ can be expressed by

$$F(\xi) = a + (1 + \xi)^\alpha \sum_{m=0}^{+\infty} \sum_{n=1}^{+\infty} b_{m,n} (1 + \xi)^{m(\alpha - n)}, \quad (11)$$

where a and $b_{m,n}$ are coefficients. This provides us with *the rule of solution expression*.

2.3. Choosing initial guess and auxiliary linear operator

Due to the boundary conditions (6) and the rule of solution expression described by (11), it is straightforward to choose

$$F_0(\xi) = \frac{(1 + \xi)^\alpha - 1}{\alpha} \quad (12)$$

as the initial approximation of $F(\xi)$. Furthermore, due to the boundary conditions (6) and the foregoing rule of solution expression, it is natural to choose the auxiliary linear operator

$$\begin{aligned} \mathcal{L}[\Phi(\xi; q)] = & (1 + \xi)^3 \frac{\partial^3 \Phi(\xi; q)}{\partial \xi^3} + 2(1 - \alpha)(1 + \xi)^2 \frac{\partial^2 \Phi(\xi; q)}{\partial \xi^2} \\ & - \alpha(1 - \alpha)(1 + \xi) \frac{\partial \Phi(\xi; q)}{\partial \xi} \end{aligned} \quad (13)$$

with the property

$$\mathcal{L}[C_0 + C_1(1 + \xi)^\alpha + C_2(1 + \xi)^{\alpha+1}] = 0, \tag{14}$$

where α is defined by (8) and C_0, C_1, C_2 are coefficients.

2.4. The zero-order deformation equation

Due to the governing Eq. (5) we define the nonlinear operator

$$\mathcal{N}[\Phi(\xi; q)] = \frac{\partial^3 \Phi(\xi; q)}{\partial \xi^3} + (\kappa - 1)\Phi(\xi; q) \frac{\partial^2 \Phi(\xi; q)}{\partial \xi^2} - 2\kappa \left[\frac{\partial \Phi(\xi; q)}{\partial \xi} \right]^2. \tag{15}$$

Let \hbar denote a nonzero auxiliary parameter and

$$H(\xi) = (1 + \xi)^\gamma, \tag{16}$$

an auxiliary function, where γ is a real number to be determined later. Then, we construct the zero-order deformation equation

$$(1 - q)\mathcal{L}[\Phi(\xi; q) - F_0(\xi)] = \hbar q H(\xi) \mathcal{N}[\Phi(\xi; q)], \tag{17}$$

subject to the boundary conditions

$$\Phi(0; q) = 0, \quad \left. \frac{\partial \Phi(\xi; q)}{\partial \xi} \right|_{\xi=0} = 0, \quad \left. \frac{\partial \Phi(\xi; q)}{\partial \xi} \right|_{\xi=+\infty} = 0, \tag{18}$$

where $q \in [0, 1]$ is an embedding parameter. When $q = 0$, it is straightforward that

$$\Phi(\xi; 0) = F_0(\xi). \tag{19}$$

When $q = 1$, the zero-order deformation equations (17) and (18) are equivalent to the original Eqs. (5) and (6) so that

$$\Phi(\xi; 1) = F(\xi). \tag{20}$$

So, as the embedding parameter q increases from 0 to 1, $\Phi(\xi; q)$ varies (or deforms) from the initial approximation $F_0(\xi)$ to the solution $F(\xi)$ of the original Eqs. (5) and (6).

Due to Taylor’s theorem and (19), we expand $\Phi(\xi; q)$ in the power series

$$\Phi(\xi; q) \sim F_0(\xi) + \sum_{m=1}^{+\infty} F_m(\xi) q^m, \tag{21}$$

where

$$F_m(\xi) = \frac{1}{m!} \left. \frac{\partial^m \Phi(\xi; q)}{\partial q^m} \right|_{q=0}. \quad (22)$$

Assume that the above series is convergent when $q = 1$, we have due to (20) that

$$F(\xi) = F_0(\xi) + \sum_{m=1}^{+\infty} F_m(\xi). \quad (23)$$

2.5. The high-order deformation equation

Differentiating the zero-order deformation equations (17) and (18) m times with respect to q and then dividing them by $m!$ and finally setting $q = 0$, we have the so-called m th-order deformation equation

$$\mathcal{L}[F_m(\xi) - \chi_m F_{m-1}(\xi)] = \hbar H(\xi) R_m(\xi), \quad (24)$$

subject to the boundary conditions

$$F_m(0) = F'_m(0) = F'_m(+\infty) = 0, \quad (25)$$

where

$$\begin{aligned} R_m(\xi) &= \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}[\Phi(\xi; q)]}{\partial q^{m-1}} \right|_{q=0} \\ &= F'''_{m-1}(\xi) + \sum_{n=0}^{m-1} [(\kappa-1)F_n(\xi)F''_{m-1-n}(\xi) - 2\kappa F'_n(\xi)F'_{m-1-n}(\xi)] \end{aligned} \quad (26)$$

and

$$\chi_m = \begin{cases} 0, & \text{when } m \leq 1, \\ 1, & \text{when } m \geq 2. \end{cases} \quad (27)$$

2.6. Rule of coefficient ergodicity

The m th-order deformation equations (24) and (25) are *linear* and thus can be easily solved, especially by means of symbolic computation software such as Mathematica, Maple, MathLab and so on. The value of γ in the auxiliary function $H(\xi)$ defined by (16) is determined by *both* of the foregoing rule of solution expression and the so-called *rule of coefficient ergodicity*, i.e. *all coefficients of the solution can be modified when the order of approximation tends to infinity*. Due to the rule of solution expression described by (11), γ should be an integer. It is found that when $\gamma \geq 2$ the solution contains the terms $(1 + \xi)^{\gamma+1}$

so that the rule of solution expression is disobeyed. When $\gamma \leq 0$ the coefficients of some terms such as $(1 + \xi)^{\alpha-1}$ are always zero and thus can not be improved even as the order of approximation tends to infinity. This however disobeys the rule of coefficient ergodicity. Thus, $\gamma = 1$ and therefore the auxiliary function

$$H(\xi) = 1 + \xi \tag{28}$$

is uniquely determined by both of the rule of solution expression and the rule of coefficient ergodicity.

2.7. Recursive expression of solution

Using the auxiliary function (28) and solving first several m th-order deformation equations (24) and (25) for $m = 1, 2, 3, \dots$, we find that $F_m(\xi)$ can be expressed in general by

$$F_m(\xi) = A_m + (1 + \xi)^\alpha \sum_{i=0}^m \sum_{j=i}^{2m-i} B_m^{i,j} (1 + \xi)^{i\alpha-j}, \tag{29}$$

where A_m and $B_m^{i,j}$ are coefficients. To ensure that the above expression holds for all $m \geq 1$, we substitute it into the m th-order deformation equation (24) and obtain the following recurrence formula

$$B_m^{i,j} = \chi_m \chi_{m+1-i} \chi_{2m-i-j} B_{m-1}^{i,j} + \frac{\hbar \gamma_m^{i,j}}{(i\alpha - j)(i\alpha - j - 1)[(i + 1)\alpha - j]} \tag{30}$$

for $0 \leq i \leq m$, $i \leq j \leq 2m - i$ when $i^2 + j^2 \neq 0$ and

$$(i\alpha - j)(i\alpha - j - 1)[(i + 1)\alpha - j] \neq 0,$$

where

$$\begin{aligned} \gamma_m^{i,j} &= \chi_{m+1-i} \chi_{j-i} [(i + 1)\alpha - j][(i + 1)\alpha - j + 1][(i + 1)\alpha - j + 2] B_{m-1}^{i,j-2} \\ &+ (\kappa - 1)(\chi_{m+1-i} \chi_{j+1-i} \chi_{2m+1-i-j} \beta_m^{i,j} + \chi_{i+1} \Delta_m^{i,j}) - 2\kappa \chi_{i+1} \delta_m^{i,j} \end{aligned} \tag{31}$$

with the definitions

$$\begin{aligned} \delta_m^{i,j} &= \sum_{n=0}^{m-1} \sum_{r=\max\{0, i+n-m\}}^{\min\{n, i-1\}} \sum_{s=\max\{r, i+j+2n-2m-r\}}^{\min\{2n-r, r+j-i\}} [(r + 1)\alpha - s] \\ &\times [(i - r)\alpha - j + s + 1] B_n^{r,s} B_{m-1-n}^{i-r-1, j-s-1}, \end{aligned} \tag{32}$$

$$\begin{aligned} \Delta_m^{i,j} &= \sum_{n=0}^{m-1} \sum_{r=\max\{0, i+n-m\}}^{\min\{n, i-1\}} \sum_{s=\max\{r, i+j+2n-2m-r\}}^{\min\{2n-r, r+j-i\}} \\ &\times [(i - r)\alpha - j + s][(i - r)\alpha - j + s + 1] B_n^{r,s} B_{m-1-n}^{i-r-1, j-s-1}, \end{aligned} \tag{33}$$

$$\beta_m^{i,j} = \sum_{n=0}^{\min\{m-1-i, Q(2m-i-j-1)/2\}} [(i+1)\alpha - j][(i+1)\alpha - j + 1] A_n B_{m-1-n}^{i,j-1} \quad (34)$$

and

$$Q(2k) = Q(2k+1) = 2k \quad (35)$$

for integers $k \geq 0$. Besides,

$$B_m^{i,j} = 0 \quad \text{when } (i\alpha - j)(i\alpha - j - 1)[(i+1)\alpha - j] = 0. \quad (36)$$

Due to the boundary condition (25), we have

$$B_m^{0,0} = - \sum_{j=1}^{2m} \left(1 - \frac{j}{\alpha}\right) B_m^{0,j} - \sum_{i=1}^m \sum_{j=i}^{2m-i} \left(i + 1 - \frac{j}{\alpha}\right) B_m^{i,j}, \quad (37)$$

$$A_m = - \sum_{i=0}^m \sum_{j=i}^{2m-i} B_m^{i,j}. \quad (38)$$

The first two coefficients

$$A_0 = -\frac{1}{\alpha}, \quad B_0^{0,0} = \frac{1}{\alpha} \quad (39)$$

are given by the initial guess (12). Thus, using these two coefficients and foregoing recursive formulas (30)–(38), we can successively calculate $F_m(\zeta)$ for $m = 1, 2, 3, \dots$. At the M th-order of approximation, we have

$$F(\zeta) \approx \sum_{m=0}^M \left[A_m + (1 + \zeta)^\alpha \sum_{i=0}^m \sum_{j=i}^{2m-i} B_m^{i,j} (1 + \zeta)^{i\alpha - j} \right]. \quad (40)$$

Note that the coefficients of above expression are dependent upon the auxiliary parameter \hbar . Assuming that \hbar is so properly chosen that the above series converges, we have the *explicit* analytic solution

$$F(\zeta) = \lim_{M \rightarrow +\infty} \sum_{m=0}^M \left[A_m + (1 + \zeta)^\alpha \sum_{i=0}^m \sum_{j=i}^{2m-i} B_m^{i,j} (1 + \zeta)^{i\alpha - j} \right]. \quad (41)$$

2.8. Convergence theorem

As proved by Liao [7] in general, if \hbar is properly chosen so that the series (21) is convergent at $q = 1$, one can get as accurate approximations as possible by means of the series (23). Similarly, we have

Theorem 1 (Convergence theorem). *The series (23) is an exact solution of Eqs. (5) and (6) as long as it is convergent.*

Proof. Due to the definition (27) of χ_m and the m th-order deformation equation (24), it holds

$$\hbar H(\xi) \sum_{m=1}^M R_m(\xi) = \sum_{m=1}^M \mathcal{L}[F_m(\xi) - \chi_m F_{m-1}(\xi)] = \mathcal{L}[F_M(\xi)].$$

If the series (23) is convergent, it must hold

$$\lim_{M \rightarrow +\infty} F_M(\xi) = 0.$$

Thus, due to the definition (13) of \mathcal{L} and above two expressions, we have

$$\hbar H(\xi) \sum_{m=1}^{+\infty} R_m(\xi) = \lim_{M \rightarrow +\infty} \mathcal{L}[F_M(\xi)] = \mathcal{L} \left[\lim_{M \rightarrow +\infty} F_M(\xi) \right] = 0,$$

which gives

$$\sum_{m=1}^{+\infty} R_m(\xi) = 0$$

because both of the auxiliary parameter \hbar and the auxiliary function $H(\xi)$ defined by (28) are nonzero. Substituting the definition (26) of $R_m(\xi)$ into above expression, we have

$$\begin{aligned} \sum_{m=1}^{+\infty} R_m(\xi) &= \frac{d^3}{d\xi^3} \left[\sum_{m=0}^{+\infty} F_m(\xi) \right] + (\kappa - 1) \left[\sum_{m=0}^{+\infty} F_m(\xi) \right] \frac{d^2}{d\xi^2} \left[\sum_{m=0}^{+\infty} F_m(\xi) \right] \\ &\quad - 2\kappa \frac{d}{d\xi} \left[\sum_{m=0}^{+\infty} F_m(\xi) \right] \frac{d}{d\xi} \left[\sum_{m=0}^{+\infty} F_m(\xi) \right] = 0. \end{aligned} \tag{42}$$

Besides, using the boundary conditions (25) and the definition (12) of the initial guess $F_0(\xi)$, we have

$$\sum_{m=0}^{+\infty} F_m(0) = 0, \quad \sum_{m=0}^{+\infty} F'_m(0) = 1, \quad \sum_{m=0}^{+\infty} F'_m(+\infty) = 0. \tag{43}$$

Thus, due to (42) and (43), the series

$$\sum_{m=0}^{+\infty} F_m(\xi)$$

must be an exact solution of equations (5) and (6). This ends the proof. \square

2.9. Determining the region of \hbar for validity

Note that the solution (41) contains the auxiliary parameter \hbar , which we have great freedom to choose. The validity of foregoing analytic approach is based on such an assumption that the series (21) converges at $q = 1$. It is the

auxiliary parameter \hbar which ensures that this assumption can be satisfied, as verified in our previous publications [7,9–13]. Generally, for any an analytic solution given by the homotopy analysis method, one should provide the corresponding region of \hbar , in which the given analytic solution is valid.

When $\kappa = 1/3$ there exists an exact solution

$$F(\xi) = -3 \left(\frac{c^2}{9} \right)^{1/6} \frac{Ai'(z)}{Ai(z)}, \quad z = \frac{(c\xi - 1)}{3} \left(\frac{9}{c^2} \right)^{1/3}, \quad (44)$$

where $Ai(z)$ is Airy function and $c = -F''(0) = 0.56144919346$, as reported by Kuiken [14]. Obviously, this exact analytic solution can be employed to verify the validity of the proposed analytic approach. It is found that the series (41) is convergent when $-1 \leq \hbar < 0$ and $\kappa = 1/3$. When $\kappa = 1/3$ and $\hbar = -1$, our 20th-order approximation agrees well with the exact solution (44), as shown in Fig. 1. And the corresponding value of $F''(0)$ converges to the exact one $F''(0) = -0.56144919346$, as shown in Table 1. This clearly indicates the validity of our analytic approach. Furthermore, it is found that when $-1 \leq \hbar < 0$ the series (41) is valid in the *whole* region $0 < \kappa < 1$, as shown in Fig. 2. Thus, the explicit analytic solution (41) when $\hbar = -1$ can be regarded as a kind of *definition* of the nonlinear equations (5) and (6). Note that, different from Kuiken's asymptotic expression (7), the solution (41) is a *purely* analytic solution and is valid in the *whole* region $0 \leq \xi \leq +\infty$.

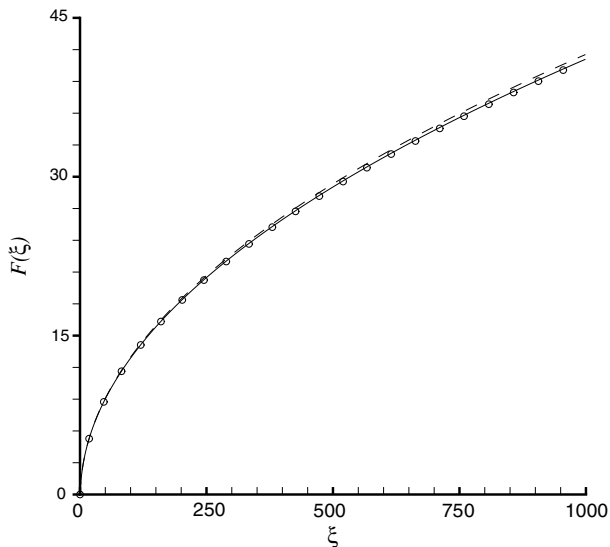


Fig. 1. Comparison of the exact result (44) of $F(\xi)$ with the analytic approximations (40) given by the homotopy analysis method when $\kappa = 1/3$ and $\hbar = -1$. Dashed line: 10th-order HAM approximation; solid line: 20th-order HAM approximation; circle: exact solution (44).

Table 1
The value of $F''(0)$ when $\kappa = 1/3$ at the n th-order of approximation given by the homotopy analysis method with $\hbar = -1$

n	$F''(0)$ given by the homotopy analysis method
5	-0.56360
10	-0.56158
15	-0.56139
20	-0.56141
25	-0.56143
30	-0.56144
35	-0.56145
40	-0.56145

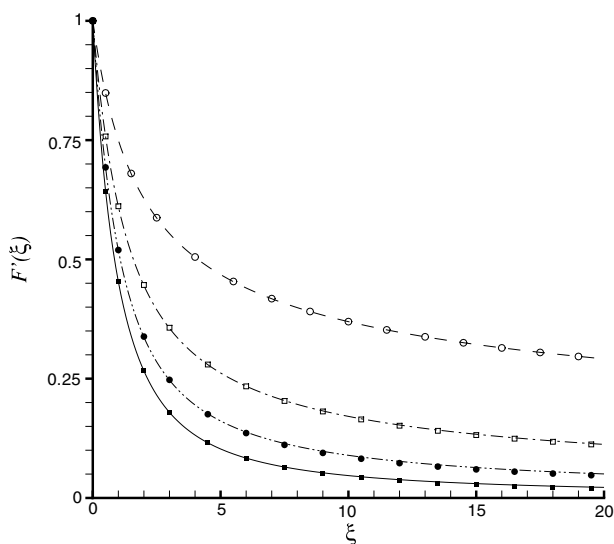


Fig. 2. Comparison of the numerical results of $F''(\xi)$ with the analytic approximations (40) given by the homotopy analysis method when $\hbar = -1$. Dashed line: 10th-order HAM approximation when $\kappa = 1/5$; dash-dotted line: 10th-order HAM approximation when $\kappa = 2/5$; dash-dot-dotted line: 10th-order HAM approximation when $\kappa = 3/5$; solid line: 20th-order HAM approximation when $\kappa = 4/5$; circle: numerical solution when $\kappa = 1/5$; square: numerical solution when $\kappa = 2/5$; filled circle: numerical solution when $\kappa = 3/5$; filled square: numerical solution when $\kappa = 4/5$.

3. Homotopy-Padé approach

As verified in our previous publications [7,9–13], it is the auxiliary parameter \hbar which provides us with a simple way to adjust or control the convergence rate and region of approximations given by the homotopy analysis method. Alternatively, in many (but not all) cases the convergence rate and/or region of

approximations given by the homotopy analysis method can be greatly enlarged by the so-called Homotopy-Padé approach proposed by Liao and Cheung [13]. To explain it, consider the series

$$\Phi(\xi; q) \sim F_0(\xi) + \sum_{m=1}^{2n} F_m(\xi)q^m. \quad (45)$$

Applying the traditional $[n, n]$ Padé approximant to above power series of q , we have

$$\Phi(\xi; q) \sim \frac{\sum_{i=0}^n \epsilon_{n,i}(\xi)q^i}{\sum_{i=0}^n \mu_{n,i}(\xi)q^i}. \quad (46)$$

Setting $q = 1$ in above expression, we have due to (20) that

$$F(\xi) \approx \frac{\sum_{i=0}^n \epsilon_{n,i}(\xi)}{\sum_{i=0}^n \mu_{n,i}(\xi)}. \quad (47)$$

Different from the traditional Padé approximant, the functions $\epsilon_{n,i}(\xi)$ and $\mu_{n,i}(\xi)$ are *not* necessary to be power functions of ξ at all. Similarly, employing the traditional $[n, n]$ Padé approximant to the series

$$\left. \frac{\partial^2 \Phi(\xi; q)}{\partial \xi^2} \right|_{\xi=0} \sim F_0''(0) + \sum_{m=1}^{2n} F_m''(0)q^m, \quad (48)$$

we have

$$\left. \frac{\partial^2 \Phi(\xi; q)}{\partial \xi^2} \right|_{\xi=0} \sim \frac{\sum_{i=0}^n \sigma_{n,i}q^i}{1 + \sum_{i=1}^n \rho_{n,i}q^i}, \quad (49)$$

which gives due to (20) that

$$F''(0) \approx \frac{\sum_{i=0}^n \sigma_{n,i}}{1 + \sum_{i=1}^n \rho_{n,i}} \quad (50)$$

by setting $q = 1$. It is interesting that all of the functions $\epsilon_{n,i}(\xi)$, $\mu_{n,i}(\xi)$ and the constants $\sigma_{n,i}$, $\rho_{n,i}$ are *independent* upon the auxiliary parameter \hbar . Thus, the results given by the homotopy-Padé approach are *independent* upon the auxiliary parameter \hbar . Besides, it is found that the homotopy-Padé approximant converges faster than the traditional Padé approximant. The same qualitative conclusions were reported by Liao and Cheung [13].

It is found that when $\kappa = 1/3$ the $[n, n]$ homotopy-Padé approximant of $F''(0)$ converges quickly to the exact value $F''(0) = -0.56144919$, as shown in Table 2. Besides, the corresponding $[5, 5]$ homotopy-Padé approximant of $F(\xi)$ is more accurate than the 10th-order approximation and agrees well with the exact solution (44), as shown in Fig. 3. Furthermore, it is found that the homotopy-Padé approaches mentioned above are valid for the whole $0 < \kappa < 1$

Table 2
The $[n, n]$ homotopy-Padé approximant of $F''(0)$ when $\kappa = 1/3$

n	Homotopy-Padé approximant of $F''(0)$
2	-0.5609771
4	-0.5613269
6	-0.5616108
8	-0.5614565
10	-0.5614483
12	-0.5614489
14	-0.5614449
16	-0.56144923
17	-0.56144921
18	-0.56144923
19	-0.56144919
20	-0.56144919

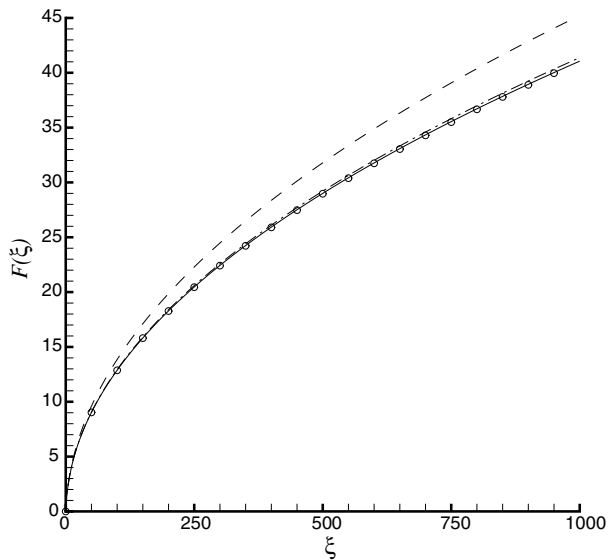


Fig. 3. Comparison of the exact solution (44) of $F(\xi)$ with the homotopy-Padé approximation (47) when $\kappa = 1/3$. Dashed line: $[1, 1]$ homotopy-Padé approximation; dash-dotted line: $[3, 3]$ homotopy-Padé approximation; solid line: $[5, 5]$ homotopy-Padé approximation; circle: exact solution (44).

and the corresponding series of $F(\xi)$ and $F''(0)$ converge rather quickly. The convergent analytic results of $F''(0)$ are listed in Table 3, which agree quite well with Kuiken’s numerical results [14].

Table 3

The convergent analytic values of $F''(0)$ given by the homotopy analysis method

κ	$F''(0)$
0.1	-0.215052
0.2	-0.381913
0.3	-0.519994
0.4	-0.638989
0.5	-0.744394
0.6	-0.839613
0.7	-0.926891
0.8	-1.007792
0.9	-1.083447

4. Conclusions

In this paper a powerful, easy-to-use analytic technique for nonlinear problems in general, namely the homotopy analysis method, is further improved and systematically described through a typical nonlinear problem, i.e. the viscous boundary layer flow due to a moving sheet, governed by (2) and (3). A usual procedure of the homotopy analysis method is proposed for the first time. Two rules, the rule of solution expression and the rule of coefficient ergodicity, are proposed, which play important roles in the frame of the homotopy analysis method and simplify its applications in science and engineering. An explicit analytic solution (41) of considered nonlinear problem is given for the first time, with recursive formulas (30)–(39) for coefficients. This analytic solution agrees well with numerical results and can be regarded (when $-1 \leq \hbar < 0$) as a definition of the solution of the nonlinear equations (5) and (6).

This paper shows us the validity and great potential of the homotopy analysis method for nonlinear problems in science and engineering.

Acknowledgements

Thanks to “National Science Fund for Distinguished Young Scholars” (Approval no. 50125923) of Natural Science Foundation of China for the financial support.

References

- [1] J.D. Cole, *Perturbation Methods in Applied Mathematics*, Blaisdell Publishing Company, Waltham Massachusetts, 1968.
- [2] Ali Hasan Nayfeh, *Perturbation Methods*, John Wiley and Sons, New York, 2000.

- [3] A.M. Lyapunov, *General Problem on Stability of Motion* (English translation), Taylor and Francis, London, 1992 (Original work published 1892).
- [4] A.V. Karmishin, A.I. Zhukov, V.G. Kolosov, *Methods of Dynamics Calculation and Testing for Thin-Walled Structures*, Mashinostroyeniye, Moscow, 1990 (in Russian).
- [5] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston and London, 1994.
- [6] S.J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems*, PhD thesis, Shanghai Jiao Tong University, 1992.
- [7] S.J. Liao, An explicit, totally analytic approximation of Blasius viscous flow problems, *Int. J. Non-Linear Mech.* 34 (4) (1999) 759–778.
- [8] S.J. Liao, A simple way to enlarge the convergence region of perturbation approximations, *Int. J. Non-linear Dynam.* 19 (2) (1999) 93–110.
- [9] S.J. Liao, A uniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate, *J. Fluid Mech.* 385 (1999) 101–128.
- [10] S.J. Liao, A. Campo, Analytic solutions of the temperature distribution in Blasius viscous flow problems, *J. Fluid Mech.* 453 (2002) 411–425.
- [11] S.J. Liao, An analytic approximation of the drag coefficient for the viscous flow past a sphere, *Int. J. Non-Linear Mech.* 37 (2002) 1–18.
- [12] S.J. Liao, An explicit analytic solution to the Thomas–Fermi equation, *Appl. Math. Comput.* 144 (2003) 433–444.
- [13] S.J. Liao, K.F. Cheung, Analytic solution for nonlinear progressive waves in deep water, *J. Engrg. Math.*, in press.
- [14] H.K. Kuiken, On boundary layers in fluid mechanics that decay algebraically along stretches of wall that are not vanishingly small, *IMA J. Appl. Math.* 27 (1981) 387–405.