A Simple User Guide

The NOPH 1.0 is a Maple package for highly periodically oscillating system governed by

$$\ddot{U}_i(t) = f_i[\mathbf{U}(t), \dot{\mathbf{U}}(t), \ddot{\mathbf{U}}(t)], \qquad 1 \le i \le \kappa, \tag{1}$$

where the dependent variable $\mathbf{U}(t)$ has κ components $U_1(t), \ldots, U_\kappa(t)$, the independent variable t denotes time, the dot denotes differentiation with respect to t, and $f_i[\mathbf{U}(t), \dot{\mathbf{U}}(t), \ddot{\mathbf{U}}(t)]$ is either an algebraic or a rational function of $\mathbf{U}(t), \dot{\mathbf{U}}(t)$ and $\ddot{\mathbf{U}}(t)$. Note that Eqs. (1) are rather general which can be used to describe lots of periodic oscillations of nonlinear problems in science and engineering. In the package NOPH, we mainly focus on single conservative oscillations and self-excited oscillations. The former are corresponding to periodic oscillations of center type, and the latter to periodic oscillations of limit cycle type, respectively.

To load the package NOPH, one can proceed as follows:

$> restart:$ initializing Maple.

$> currentdir(\text{“}C:/oscillation\text{”});$ setting current work directory.

Please note, the program and sample files should be located into current directory.

$> read\text{‘}period\_conser\_self.mpl\text{‘}:$ reading the program file into memory

$> with(NOPH);$ loading the package NOPH.

In what follows two examples are given to show how to use the package NOPH.

1. Example 1

Consider Duffing oscillation equation

$$\ddot{U} = -U + 2U^3 \tag{2}$$

subject to the initial condition

$$U(0) = 1/2, \ \ \dot{U}(0) = 0. \tag{3}$$

It is a free oscillating system with just odd nonlinearity.

To solve the system (2) and (3), one can run the main procedure as follows:

$> main([diff(U(t), t\$2) = -U(t) + 2 * U(t)^3], [U(0) = 1/2, D(U)(0) = 0], 5, 8);$

The package NOPH delivered the following results:

The input equation and initial conditions are:

$$\frac{\partial^2}{\partial t^2}U(t) + U(t) - 2U^3(t),$$
$$U(0) = 1/2, D(U)(0) = 0.$$

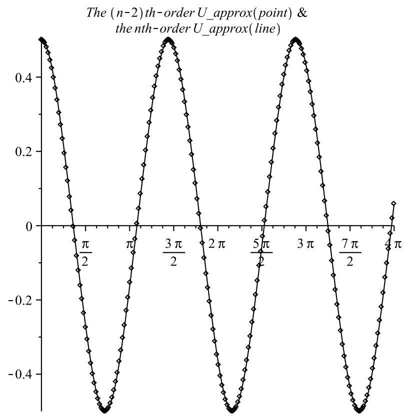
The new EQ and initial conditions in u with the form:

$$w^2 \frac{\partial^2}{\partial \tau^2}u(\tau) + u(\tau) - 2u^3(\tau),$$
$$u(0) = 1/2, D(u)(0) = 0,$$

where:

$$\tau = \omega t, \quad U(t) = u(\tau).$$

It take about 3.797 seconds to complete the first part computation.
The optimal value of h is: $\{h = -1.0\}$. Where: $n = 8$.



The approximations of ω and the corresponding errors:

i	ω_i	errors
1	0.7905	$---$
2	0.7846	0.0075
3	0.7846	0.0000
4	0.7845	0.0000
5	0.7845	0.0000
6	0.7845	0.0000

The approximants of $\omega_{Padé}$ and the corresponding errors:

i	$[Padé]\omega_i$	errors
1	0.7845	$---$
2	0.7845	0.0000
3	0.7845	0.0000

It takes about 0.890 seconds to complete the second part computation.
The total time is:10.828 seconds.

2. Example 2

Consider a two directly coupled van der Pol oscillator

$$\begin{cases} \ddot{U}(t) - \varepsilon(1 - U^2(t))\dot{U}(t) + U(t) = \varepsilon\mu(V(t) - U(t)), \\ \ddot{V}(t) - \varepsilon(1 - V^2(t))\dot{V}(t) + p^2V(t) = \varepsilon\mu(U(t) - V(t)), \end{cases} \tag{4}$$

with initial conditions

$$U(0) = a_1, \quad \dot{U}(0) = b_1, \quad V(0) = c_1, \quad \dot{V}(0) = 0, \tag{5}$$

where ε, μ, p are unknown physical parameters.

This system is a coupled self excited system. We solve this system by considering three different cases.

2.1. All physical parameters are unknown

To solve the system (4) and (5), one can run the main procedure as follows:

$> main([diff(u(t), t\$2) - epsilon * (1 - u(t)^2) * diff(u(t), t) + u(t) = epsilon * mu * (v(t) - u(t)), diff(v(t), t\$2) - epsilon * (1 - v(t)^2) * diff(v(t), t) + p^2 * v(t) = epsilon * mu * (u(t) - v(t))], [u(0) = a, D(u)(0) = b, v(0) = c, D(v)(0) = 0], 5);$

Then NOPH outputs the following algebraic equations, which are obtained by meeting the zeroth-order initial conditions:

$$-4\,\epsilon\,\omega_0 b_{1,0} - 4\,{\omega_0}^2 a_{1,0} + 4\,a_{1,0} + \epsilon\,\omega_0 {a_{1,0}}^2 b_{1,0} + 4\,\epsilon\,\mu\,a_{1,0} - 4\,\epsilon\,\mu\,c_{1,0} + \epsilon\,\omega_0 {b_{1,0}}^3 = 0,$$

$$-\,\epsilon\,\omega_0 {a_{1,0}}^3 + 4\,b_{1,0} - \epsilon\,\omega_0 a_{1,0} {b_{1,0}}^2 + 4\,\epsilon\,\omega_0 a_{1,0} + 4\,\epsilon\,\mu\,b_{1,0} - 4\,{\omega_0}^2 b_{1,0} = 0,$$

$$\epsilon\,\mu\,a_{1,0} + {\omega_0}^2 c_{1,0} - \epsilon\,\mu\,c_{1,0} - p^2 c_{1,0} = 0,$$

$$-\,\epsilon\, - 4\,\omega_0 c_{1,0} + \omega_0 {c_{1,0}}^3 + 4\,b_{1,0}\mu = 0,$$

and the variables and parameters lists:

$$[\omega_0, a_{1,0}, b_{1,0}, c_{1,0}],\ [\epsilon, \mu, p].$$

Then NOPH asks:" Please input additional conditions, {} means no additional conditions:", for this system, we input $\{b[1][0] = 0\}$ to simplify the above nonlinear algebraic equations, or else it outputs too many solutions and it is inconvenient for us to select a solution and input its order.

And then NOPH delivers the reduced nonlinear algebraic equations:

$$4\,(-{\omega_0}^2 a_{1,0} + a_{1,0} + \epsilon\,\mu\,a_{1,0} - \epsilon\,\mu\,c_{1,0}) = 0,$$

$$\epsilon\,\omega_0 a_{1,0}({a_{1,0}}^2 - 4) = 0,$$

$$\epsilon\,\mu\,a_{1,0} + {\omega_0}^2 c_{1,0} - \epsilon\,\mu\,c_{1,0} - p^2 c_{1,0} = 0,$$

$$\epsilon\,\omega_0 c_{1,0}(-4 + {c_{1,0}}^2) = 0.$$

Next, the NOPH asks: "Please input the vars and paras list to be solved:", we input $[p[1], p[2], p[3], 4, 2, 1]$, in which $p[i]$ means the ith element in the parameter list, the integer i means the ith element in the variable list. These two lists are shown in the above.

Next, NOPH outputs the variables and parameters list to be solved

$$[\epsilon, \mu, p, c_{1,0}, a_{1,0}, \omega_0],$$

and 8 group solutions for ω_0, $a_{1,0}$, $b_{1,0}$ and $c_{1,0}$:

$$\begin{aligned}
&\{p = \mp 1,\, \omega_0 = 1,\, a_{1,0} = 2,\, b_{1,0} = 0,\, c_{1,0} = 2\},\\
&\{p = \mp 1,\, \omega_0 = 1,\, a_{1,0} = -2,\, b_{1,0} = 0,\, c_{1,0} = -2\},\\
&\{p = \mp 1,\, \omega_0 = \sqrt{1 + 2\,\varepsilon\,\mu},\, a_{1,0} = 2,\, b_{1,0} = 0,\, c_{1,0} = -2\},\\
&\{p = \mp 1,\, \omega_0 = \sqrt{1 + 2\,\varepsilon\,\mu},\, a_{1,0} = -2,\, b_{1,0} = 0,\, c_{1,0} = 2\}.
\end{aligned} \tag{6}$$

[**Explain: As p appears in the form of p^2 in the Eq. (4), it is easy to verify that if $U(t)$ and $V(t)$ are solutions of the system (4) and (5), then $-U(t)$ and $-V(t)$ are also solutions for the system. Therefore, there are two essentially different solutions in (6), i.e.,** $\{\mathrm{p} = 1,\, \omega_0 = 1,\, \mathrm{a}_{1,0} = 2,\, \mathrm{b}_{1,0} = 0,\, \mathrm{c}_{1,0} = 2\}$**, and** $\{\mathrm{p} = 1,\, \omega_0 = \sqrt{1 + 2\,\varepsilon\,\mu},\, \mathrm{a}_{1,0} = -2,\, \mathrm{b}_{1,0} = 0,\, \mathrm{c}_{1,0} = 2\}$.]

Select a solution and input its order, such as 8(in (6)), our program further delivers:
The parameter constraint condition is: $\{p = 1\}$,
the [1,1] homotopy-Padé approximant

$$\omega_{P_{1,\,1}} = \frac{(\varepsilon^2 + 64\,\varepsilon\,\mu + 32)\,\sqrt{2\,\varepsilon\,\mu + 1}}{3\,\varepsilon^2 + 64\,\varepsilon\,\mu + 32}, \tag{7}$$

and the [2,2] homotopy-Padé approximant

$$\omega_{P_{2,2}} = (16640\,\varepsilon^2 + 66560\,\varepsilon^3\,\mu + 960\,\varepsilon^4 + 294912\,\varepsilon\,\mu + 66560\,\varepsilon^4\,\mu^2 + 589824\,\varepsilon^2\,\mu^2 + 49152 + 393216\,\varepsilon^3\,\mu^3 + 1920\,\varepsilon^5\,\mu + 9\,\varepsilon^6)\sqrt{2\,\varepsilon\,\mu+1}\Big/(45\,\varepsilon^6 + 393216\,\varepsilon^3\,\mu^3 + 589824\,\varepsilon^2\,\mu^2 + 78848\,\varepsilon^4\,\mu^2 + 78848\,\varepsilon^3\,\mu + 294912\,\varepsilon\,\mu + 3840\,\varepsilon^5\,\mu + 19712\,\varepsilon^2 + 49152 + 1920\,\varepsilon^4). \quad (8)$$

2.2. In the case of $\varepsilon = 1/2$

From (6), we know that there is parameter constraint $p = 1$ in each solution. Therefore, letting $\varepsilon = 1/2$, there will be only one unknown parameter μ contained in input system. In this case, one can run the main procedure as follows:

$> main([diff(u(t), t\$2) - 1/2 * (1 - u(t)^2) * diff(u(t), t) + u(t) = 1/2 * mu * (v(t) - u(t)), diff(v(t), t\$2) - 1/2 * (1 - v(t)^2) * diff(v(t), t) + p^2 * v(t) = 1/2 * mu * (u(t) - v(t))], [u(0) = a, D(u)(0) = b, v(0) = c, D(v)(0) = 0], 5);$

Following the similar steps shown in section 2.1, our program automatically delivers (by selecting the 8th solution in (6) and setting $h\_order = 5$) the following results within 590 seconds.
The parameter constraint condition is: $\{p = 1\}$,
the [1,1] homotopy-Padé approximant

$$\omega_{P_{1,1}} = \frac{(128\,\mu + 129)\,\sqrt{\mu+1}}{128\,\mu + 131}, \quad (9)$$

and the [2,2] homotopy-Padé approximant

$$\omega_{P_{2,2}} = \frac{(9973504\,\mu + 9703424\,\mu^2 + 3415817 + 3145728\,\mu^3)\,\sqrt{\mu+1}}{3468845 + 3145728\,\mu^3 + 9752576\,\mu^2 + 10075648\,\mu}. \quad (10)$$

Moreover, our program also provides a figure to compare homotopy-Padé approximants of ω at different orders, as shown in Fig.1. It indicates that $[1,1]$ and $[2,2]$ homotopy-Padé approximants of ω agree very well.

2.3. In the case of $\varepsilon = 1/2,\ \ \mu = 8$

In this case, there is no unknown parameter in input system, for there is parameter constraint $p = 1$ involved in each solution. One can run the main procedure as follows:

$> main([diff(u(t), t\$2) - 1/2 * (1 - u(t)^2) * diff(u(t), t) + u(t) = 4 * (v(t) - u(t)), diff(v(t), t\$2) - 1/2 * (1 - v(t)^2) * diff(v(t), t) + p^2 * v(t) = 4 * (u(t) - v(t))], [u(0) = a, D(u)(0) = b, v(0) = c, D(v)(0) = 0], 7, 9);$

Following the similar steps shown in Section 2.1, our program automatically outputs (by setting $h\_order = 7$, $sol\_order = 9$) $\hbar = -0.498$, approximants of ω, a_1, b_1, c_1, as

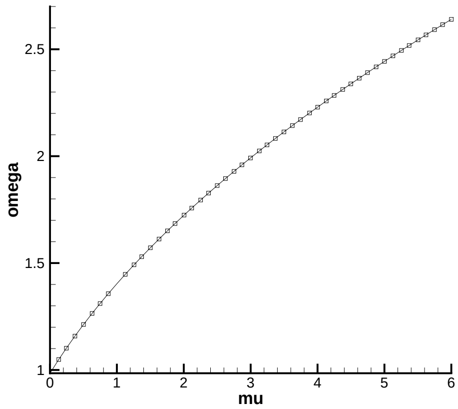


Fig. 1. The homotopy-Padé approximants of ω in Eq. (4) as a function of μ. Solid line: [1,1] homotopy-Padé approximant; Diamonds: [2,2] homotopy-Padé approximant.

well as comparison graphs for different order approximations of $U(t)$ and $V(t)$ within 755 seconds. As shown in **Fig.2**, the 7th and 9th-order HAM approximations of $U(t)$ and $V(t)$ agree very well, respectively. It is to be stressed that in **Fig.2** we also compared the HAM solutions with numerical solutions obtained by using a Fehlberg fourth-fifth order Runge-Kutta method (rkf45) with degree four interpolant. It can be seen that the 7th and 9th order HAM solutions agree very well with the corresponding imitating curves of the numerical solutions for the given values $a_1 = 1.9964$, $b_1 = -0.1234, c_1 = -1.9964$ and the absolute error 10^{-7}. **Table 1** indicates that the homotopy approximants of the frequency ω, the amplitudes a_1, b_1 and c_1 converge very quickly when setting $\hbar = -0.498$. It shows from **Table 2** that the homotopy-Padé approximants converge even faster than the corresponding homotopy approximants.

Table 1.
The HAM approximants of ω, a_1, b_1, c_1 in Eq. (4) when $\varepsilon = 1/2$, $\mu = 8$ and taking $\hbar = -0.498$.

i	ω_i	$a_{1,i}$	$b_{1,i}$	$c_{1,i}$
1	3.0000	2.0000	−0.1245	−2.0000
2	2.9964	2.0000	−0.1248	−2.0000
3	2.9964	1.9964	−0.1234	−1.9964
4	2.9964	1.9964	−0.1234	−1.9964

Table 2.
The HAM-Padé approximants of ω, a_1, b_1, c_1 in Eq. (4) when $\varepsilon = 1/2$, $\mu = 8$.

i	ω_i	$a_{1,i}$	$b_{1,i}$	$c_{1,i}$
1	2.9964	2.0000	−0.1248	−2.0000
2	2.9964	1.9964	−0.1234	−1.9964

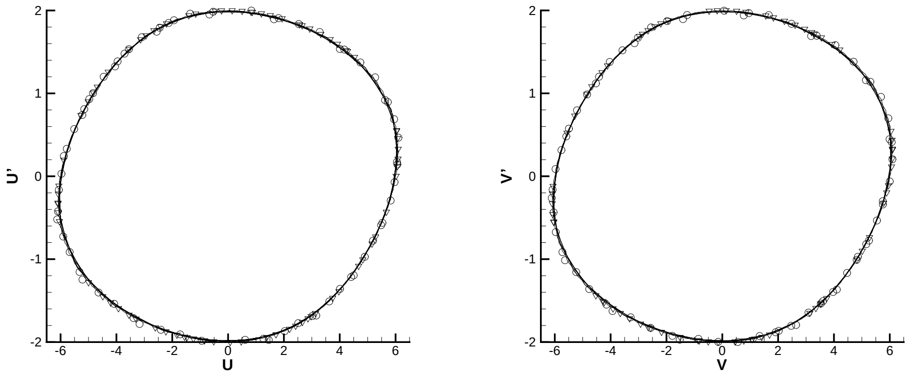


Fig. 2. Approximations of $U(t)$, $V(t)$ for Eq. (4) when $\varepsilon = 1/2$, $\mu = 8$ by means of $\hbar = -0.498$. Cycles: 7th-order HAM approximation; Gradients: 9th-order HAM approximation; Solid line: numerical solution.