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One family of 13315 stable periodic orbits of non-hierarchical unequal-mass triple systems

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The three-body problem can be traced back to Newton in 1687, but it is still an open question today. Note that only a few periodic orbits of three-body systems were found in 300 years after Newton mentioned this famous problem. Although triple systems are common in astronomy, practically all observed periodic triple systems are hierarchical (similar to the Sun, Earth and Moon). It has traditionally been believed that non-hierarchical triple systems would be unstable and thus should disintegrate into a stable binary system and a single star, and consequently stable periodic orbits of non-hierarchical triple systems have been expected to be rather scarce. However, we report here one family of 135445 periodic orbits of non-hierarchical triple systems with unequal masses; 13315 among them are stable. Compared with the narrow mass range (only 10^{-5}) in which stable "Figure-eight" periodic orbits of three-body systems exist, our newly found stable periodic orbits have fairly large mass region. We find that many of these numerically found stable non-hierarchical periodic orbits have mass ratios close to those of hierarchical triple systems that have been measured with astronomical observations. This implies that these stable periodic orbits of non-hierarchical triple systems with distinctly unequal masses quite possibly can be observed in practice. Our investigation also suggests that there should exist an infinite number of stable periodic orbits of non-hierarchical triple systems with distinctly unequal masses. Note that our approach has general meaning: in a similar way, every known family of periodic orbits of three-body systems with two or three equal masses can be used as a starting point to generate thousands of new periodic orbits of triple systems with distinctly unequal masses.

three-body problem, non-hierarchical system, periodic orbits

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1 Introduction

Triple systems are common, and they are key objectives in astrophysics [1]. Although the three-body system has been investigated for more than three hundred years [2-5], it is still a challenging and open question for astrophysicists because of its inherent chaotic characteristics [6]. Recently, based on the assumption of ergodicity, Stone and Leigh [7] have ob-

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tained a statistical solution to the chaotic, non-hierarchical three-body system. It has traditionally been believed that bound, non-hierarchical triple systems are always unstable and that they disintegrate into a stable binary system and a single star [7]. Therefore, periodic orbits of the three-body problem are extremely precious since they are the only way to penetrate the fortress that was previously considered to be inaccessible [6]. However, only three families of periodic orbits had been found in more than 300 years until Šuvakov and Dmitrašinović [8] numerically found 13 distinct periodic orbits of the three-body problem with equal masses. Li and Liao [9] subsequently found more than six hundred new families of periodic orbits of the three-body system with equal masses. Li et al. [10] also obtained more than one thousand new families of periodic orbits of the three-body system with two equal-mass bodies. Among the approximately two thousand new families of periodic orbits of the three-body system, dozens of linearly stable periodic orbits were found for non-hierarchical triple systems [10, 11], however, some of them have three equal-mass bodies [11], and the others have two equal-mass bodies [10]. The famous "Figure-eight" periodic orbit [12, 13] of the equal-mass triple system is non-hierarchical and linearly stable [14]. Unfortunately, the stable mass region of the figure-eight solution is very narrow (only 10^{-5}) [15]. That is to say, the figure-eight solution is stable only when the three bodies have almost equal masses, so the probability of observing this periodic orbit is extremely low in practice. To date, non-hierarchical periodic triple stars have not been found through astronomical observations. In this paper, we focus on periodic orbits of non-hierarchical triple systems with unequal masses.

2 Numerical model and method

The motion of the Newtonian planar three-body problem is described by the differential equations:

$$\ddot{\boldsymbol{r}}_i = \sum_{j=1, j\neq i}^{3} \frac{G m_j (\boldsymbol{r}_j - \boldsymbol{r}_i)}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3}, \tag{1}$$

where m_i and $\boldsymbol{r}_i$ are mass and position of the ith body ($i = 1, 2, 3$), G is the Newtonian gravity constant, respectively. Without loss of generality, we set the gravitational constant $G = 1$ by properly choosing a characteristic mass M, a characteristic spatial length R and a characteristic time T^*.

Montgomery [16] proved that all three-body orbits of zero angular momentum have syzygies (i.e., collinear instant of three bodies) except for the Lagrange's solution. Thus, it is reasonable to consider initial conditions with the collinear configuration [17-19]. In this paper, we investigate unequal-mass triple systems with the initial positions $\boldsymbol{r}_1(0) = (x_1, 0)$, $\boldsymbol{r}_2(0) = (x_2, 0)$, $\boldsymbol{r}_3(0) = (x_3, 0)$ and the initial velocities $\dot{\boldsymbol{r}}_1(0) = (0, v_1)$, $\dot{\boldsymbol{r}}_2(0) = (0, v_2)$, $\dot{\boldsymbol{r}}_3(0) = (0, v_3)$, which are perpendicular to the straight line formed by three bodies.

The first step to achieve our goal is to find periodic orbits of equal-mass triple systems with the collinear initial condition configuration mentioned above. We numerically search for periodic orbits of the three-body problem with equal masses and zero angular momentum by means of the grid search method, the Newton-Raphson method [20,21] and the numerical strategy, namely the clean numerical simulation (CNS) [22-27]. The CNS is a numerical strategy to gain reliable numerical simulation of chaotic dynamical systems, such as the three-body system. The CNS is based on an arbitrary Taylor series method [28-30] and multiple-precision arithmetic [31], plus a convergence verification by means of an additional computation with smaller numerical noise. Li and Liao [9,32] found that many periodic orbits of three-body problem might be lost by using conventional numerical algorithms in double precision. Thus, here we apply the CNS to integrate the differential equations of the three-body system.

At the beginning, we numerically search for periodic orbits of the three-body problem with equal masses $m_1 = m_2 = m_3 = 1$ and zero angular momentum. Due to the homogeneity of the potential field for the three-body problem, the initial condition x_2 can be fixed to unit. Then we choose the velocity $v_2 = -x_1 v_1$ due to zero angular momentum. Without loss of generality, we assume total momentum $m_1\dot{\boldsymbol{r}}_1 + m_2\dot{\boldsymbol{r}}_2 + m_3\dot{\boldsymbol{r}}_3 = 0$. Therefore, the initial positions can be specified as $\boldsymbol{r}_1(0) = (x_1, 0)$, $\boldsymbol{r}_2(0) = (1, 0)$, $\boldsymbol{r}_3(0) = (0, 0)$ and the initial velocities can be specified as $\dot{\boldsymbol{r}}_1(0) = (0, v_1)$, $\dot{\boldsymbol{r}}_2(0) = (0, -x_1 v_1)$, $\dot{\boldsymbol{r}}_3(0) = (0, -v_1 + x_1 v_1)$.

With the initial configuration, the orbits of the three-body problem are determined by two parameters x_1 and v_1. According to the numerical searching method of the three-body problem [8, 9], the first step is to gain approximated initial values of periodic orbits in a two-dimensional space (i.e., the x_1-v_1 plane). We investigate a region of this plane: $x_1 \in (-1, 0)$ and $v_1 \in (0, 10)$. We employ 4000×40000 uniform grid points as initial conditions in this region. With these initial conditions, the differential equations (1) are numerically solved by an eight-oder Runge Kutta ODE solver dop853 developed by Hairer et al. [33]. For each initial condition, the return proximity function $\mathrm{d}(\boldsymbol{y}(0), T_0) = \min_{t\leq T_0} \|\boldsymbol{y}(t) - \boldsymbol{y}(0)\|$ is calculated up to integration time $T_0 = 200$. We choose the initial conditions and periods T as possible candidates of periodic orbits when the return proximity function $\mathrm{d}(\boldsymbol{y}(0), T_0) < 0.1$.

The next step is to improve the precision of the approximate initial conditions of the periodic orbits using the Newton-Raphson method [20, 21] and the CNS by means of correcting the parameters x_1, v_1 and period T. The precision of the initial conditions of the periodic orbits is improved continually until the level of the return proximity function is less than 10^{-12}.

We find that one equal-mass periodic orbit has good stability. The initial condition of this periodic orbit is $\boldsymbol{r}_1(0) = (x_1, 0)$, $\boldsymbol{r}_2(0) = (1, 0)$, $\boldsymbol{r}_3(0) = (0, 0)$, $\dot{\boldsymbol{r}}_1(0) = (0, v_1)$, $\dot{\boldsymbol{r}}_2(0) = (0, v_2)$, $\dot{\boldsymbol{r}}_3(0) = (0, -(m_1v_1 + m_2v_2)/m_3)$, where $x_1 = -0.372008640907423$, $v_1 = 1.21800411067968$, $v_2 = 0.4531080538336022$ and the period $T = 7.53971451331772$ and $m_1 = m_2 = m_3 = 1$. Note that it holds $m_1v_1x_1 + m_2v_2x_2 + m_3v_3x_3 = 0$. Using the homotopy classification method [8, 34], the free group element of this periodic orbit is *bABabaBAba*. This periodic orbit has the same free group element with the moth-I orbit [8], but their periodic orbits are different. Note that, for the astrophysical three-body system, the masses of the bodies are rarely equal. Thus, using this as a starting point, we investigate periodic orbits of unequal-mass triple systems by means of the numerical continuation method [35].

The numerical continuation method is used to gain periodic solutions of a nonlinear dynamical system with a natural parameter

$$\dot{\boldsymbol{u}} = G(\boldsymbol{u}, \lambda). \tag{2}$$

Using a known periodic orbit $\boldsymbol{u}_0$ at λ_0 as initial guess, we can obtain a new periodic orbit $\boldsymbol{u}'$ at $\lambda + \Delta\lambda$ by means of the Newton-Raphson method [20, 21] and the CNS [22-27] when $\Delta\lambda$ is sufficient small to guarantee the convergence of iteration.

Because of homogeneity of the potential field of the three-body problem, we can fix the initial distance of two bodies to unit. Without loss of generality, we consider the case of zero momentum (i.e., $m_1\dot{\boldsymbol{r}}_1 + m_2\dot{\boldsymbol{r}}_2 + m_3\dot{\boldsymbol{r}}_3 = 0$). The periodic orbits are determined by x_1, v_1, v_2 and T with masses m_1, m_2 and m_3. Therefore, the initial positions of three bodies can be described by

$$\boldsymbol{r}_1(0) = (x_1, 0), \quad \boldsymbol{r}_2(0) = (1, 0), \quad \boldsymbol{r}_3(0) = (0, 0), \tag{3}$$

and the initial velocities can be described by

$$\begin{aligned} &\dot{\boldsymbol{r}}_1(0) = (0, v_1), \quad \dot{\boldsymbol{r}}_2(0) = (0, v_2), \\ &\dot{\boldsymbol{r}}_3(0) = \left(0, -\frac{m_1v_1 + m_2v_2}{m_3}\right). \end{aligned} \tag{4}$$

With the fixed masses $m_2 = m_3 = 1$, periodic orbits can be obtained by means of the numerical continuation method for different masses m_1. Using a periodic orbit with equal masses as a starting point, we apply the Newton-Raphson method and the CNS to gain a new periodic orbit at $m_1 + \Delta m$ by continually modifying the parameters x_1, v_1, v_2 and T, where Δm is small enough to guarantee the convergence of iteration. In this way, we can gain periodic orbits with different masses $m_1 \neq 1$ and $m_2 = m_3 = 1$.

Similarly, using the above periodic orbits with $m_1 \neq 1$ and $m_2 = m_3 = 1$ as starting points, we further employ the Newton-Raphson method and the CNS to gain periodic orbits at $m_2 + \Delta m$ by continuously correcting the parameters x_1, v_1, v_2 and T, where Δm is small enough to guarantee the convergence of iteration. Consequently, we gain periodic orbits of the triple system with unequal masses $m_1 \neq m_2 \neq m_3$.

Note that the periodic orbits might have nonzero angular momentum since we do not restrict the angular momentum.

3 Numerical results

Starting from the periodic orbit of the equal-mass triple system mentioned above, we obtain 135445 periodic orbits in the region of $m_1 \in [0.8, 1.1]$ and $m_2 \in [0.7, 1.2]$ with a fixed mass $m_3 = 1$ by means of the continuation method. The periodic orbits are outputted with the mass interval $\delta m = 0.001$. For the detailed initial conditions and periods, please see Supplementary Material. Three examples of these periodic orbits are shown in Figure 1. Their initial conditions and periods of the three periodic orbits are listed in Table 1.

Due to the homogeneity of the potential field for the three-body problem, there is a scaling law: $\boldsymbol{r}' = \alpha\boldsymbol{r}$, $\boldsymbol{v}' = \boldsymbol{v}/\sqrt{\alpha}$, $t' = \alpha^{3/2}t$ and energy $E' = E/\alpha$ and angular momentum $L' = \sqrt{\alpha}L$. The scale-invariant average period $\bar{T}^* = (T/k)|E|^{3/2}$ is approximately equal to a constant for periodic orbits of the three-body problem with equal masses [9, 36], where k is the number of free group words of periodic orbits. For the family of periodic orbits *bABabaBAba*, we always have the number of the free group words $k = 10$. For the newly found periodic unequal-mass orbits, Figure 2(a) shows that the scale-invariant average period $\bar{T}^* = (T/k)|E|^{3/2}$ depends on the mass of bodies. The multiple linear regression for these periodic orbits is $(T/k)|E|^{3/2} = 2.455m_1 + 1.655m_2 - 1.688$. The standard error of this multiple linear regression is 0.021. It indicates that the scale-invariant average period $\bar{T}^* = (T/k)|E|^{3/2}$ is approximately linear to m_1 and m_2 for this family of periodic orbits. Janković and Dmitrašinović [37] found that the scale-invariant angular momentum is a function of topologically rescaled period for the Broucke-Hadjidemetriou-Hénon family of periodic triple orbits with equal masses. For our newly found family of periodic orbits, it is demonstrated that the scale-invariant angular momentum $L|E|^{1/2}$ varies among different masses m_1 and m_2 as shown in Figure 2(b). It implies that the scale-invariant angular momentum also depends on the mass of bodies for this family of periodic orbits of unequal-mass triple systems. Note that some regions of the Figure 2 are blank. It suggests that no

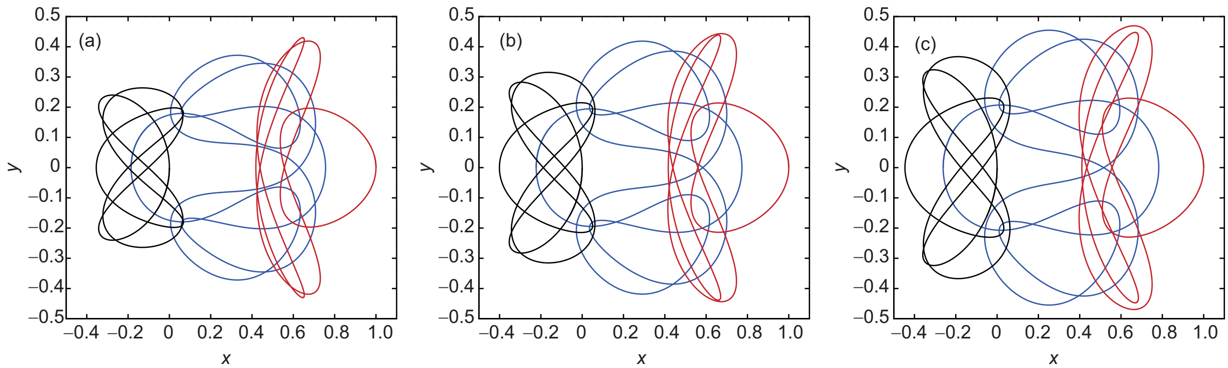

Figure 1 (Color online) Three newly found stable periodic orbits of non-hierarchical triple systems with different masses and period. (a) $m_1 = 0.87$, $m_2 = 0.8$, $m_3 = 1$ and $T = 5.9889127121$; (b) $m_1 = 0.9$, $m_2 = 0.85$, $m_3 = 1$ and $T = 6.3508660391$; (c) $m_1 = 0.93$, $m_2 = 0.89$, $m_3 = 1$ and $T = 6.6805531109$. Body-1: blue line; Body-2: red line; Body-3: black line. For their movies, please see Supplementary Material.

Table 1 Initial conditions and periods T of three stable periodic orbits for non-hierarchical three-body systems in the case of $\boldsymbol{r}_1(0) = (x_1, 0)$, $\boldsymbol{r}_2(0) = (1, 0)$, $\boldsymbol{r}_3(0) = (0, 0)$, $\dot{\boldsymbol{r}}_1(0) = (0, v_1)$, $\dot{\boldsymbol{r}}_2(0) = (0, v_2)$, $\dot{\boldsymbol{r}}_3(0) = (0, -(m_1 v_1 + m_2 v_2)/m_3)$ when $G = 1$

m_1	m_2	m_3	x_1	v_1	v_2	T
0.87	0.8	1	−0.185517464380131	2.02215468795289	0.396897646805751	5.98891271205862
0.9	0.85	1	−0.222746846934935	1.78127695164077	0.415003557019123	6.35086603914435
0.93	0.89	1	−0.261036674363779	1.58833353187897	0.430447701476608	6.68055311088189

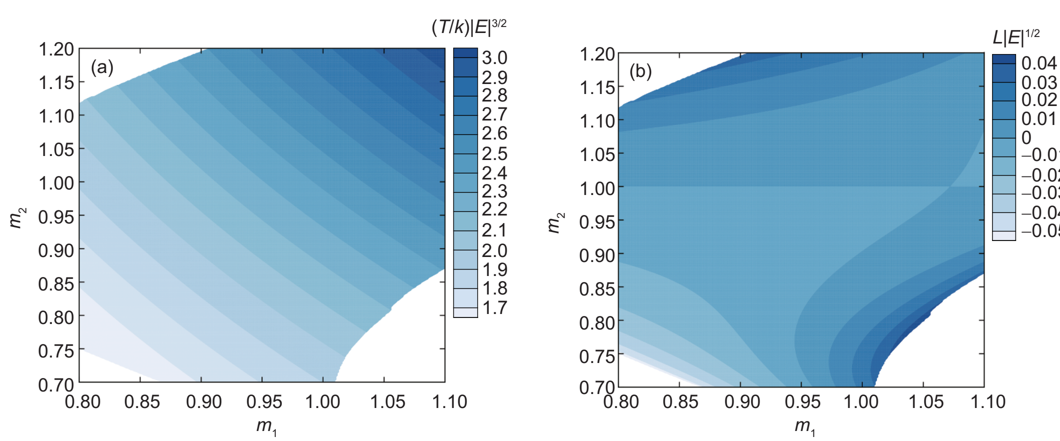

Figure 2 (Color online) The contour map of the scale-invariant average period and scale-invariant angular momentum of newly found periodic orbits. (a) The contour map of the average scale-invariant period $\bar{T}^* = (T/k)|E|^{3/2}$ in the m_1-m_2 plane, where E, T, k is total energy, period and the number of free group words of periodic orbits, respectively; (b) the contour map of the scale-invariant angular momentum $L|E|^{1/2}$ in the m_1-m_2 plane, where L is angular momentum.

periodic orbits can be found there because the orbits of the three-body system might have collision in that mass region.

Stability is an important property for periodic orbits because only stable triple systems can probably be observed. The stability of periodic orbits of the three-body system can be investigated according to the characteristic multipliers of the monondromy matrix [14]. Due to the fixed center of mass, the dimension of the planar three-body problem can be reduced to eight. We employ a theorem proved by Kepela and Simó [38] to determine the linear stability of periodic orbits of the three-body problem through the monondromy matrix. With the monodromy matrix, we can gain the equation as follows:

$$T^2 - (\alpha - 4)T + \beta - 4\alpha + 8 = 0, \tag{5}$$

where $\alpha = \mathrm{trace}(A) = \sum_{i=1}^{8} a_{ii}$, $\beta = \sum_{1\leq i<j\leq 8}(a_{ii}a_{jj} - a_{ij}a_{ji})$, a_{ij} is the elements of the monondromy matrix $\mathbf{A}$.

Therorem [38] Let T_1 and T_2 be solutions of eq. (5). If $\Delta = (\alpha - 4)^2 - 4(\beta - 4\alpha + 8) > 0$, $|T_1| < 2$ and $|T_2| < 2$, then all eigenvalues of the monodromy matrix $\mathbf{A}$ are on the unit circle.

Using this theorem, we find that 13315 periodic orbits are linearly stable among the 135445 newly found periodic orbits. Three examples of the stable periodic orbits are shown in Table 1 and Figure 1. The domain of the masses (m_1, m_2) of stable periodic orbits is shown in Figure 3. The mass region becomes narrow when the masses m_1 and m_2 decrease. Note that the mass region of the stable figure-eight solution [12, 13] is very narrow (only 10^{-5}) [15]. So, the mass region of the newly found stable non-hierarchical periodic orbits is fairly large and their masses have apparent differences. For instance, for the stable *non-hierarchical* periodic orbit $m_1 = 0.87$, $m_2 = 0.8$ and $m_3 = 1$, we have its mass ratio $m_2/m_1 \approx 0.92$ and $m_2/m_3 = 0.8$. A recently observed *hierarchical* triple system [39] has masses 1.21, 1.14 and 1.4 $M_\odot$, corresponding to mass ratios 0.94 and 0.81. It should be emphasized that the mass ratios of our newly found stable *non-hierarchical* periodic orbits are close to the mass ratios of the hierarchical triple system which has been measured by the astronomical observation. This implies that our newly found stable *non-hierarchical* periodic orbits are likely to be observed in astronomy.

Since the dimensionless quantities are used in the above numerical results, the variables can be rescaled to applications of stellar dynamics [40] through $GMT^{*^2}/R^3 = 1$, where M, T^*, R and G are the characteristic mass, time and length and the Newtonian gravitational constant, respectively. If we choose $M = M_\odot$ and $R = 10$ AU, then $T^* = \sqrt{\frac{R^3}{GM}} \approx 5$ years. For instance, with these units of quantities, the stable *non-hierarchical* periodic orbit with $m_1 = 0.87M_\odot$, $m_2 = 0.8M_\odot$ and $m_3 = M_\odot$ has a period of about 30 years. Note that the hierarchical triple system HD 188753 has a period of 25 years and semi-major axis of 11.8 AU [41]. Thus, our newly found stable *non-hierarchical* triple systems have similar size and period with the observed *hierarchical* triple system.

4 Discussion and conclusions

There may be two reasons why non-hierarchical periodic triple stars have not yet been found through astronomical observations. On the one hand, accurate positions and motions of non-hierarchical systems are not easy to determine, because they are complicated and far away from the Earth. On the other hand, few periodic non-hierarchical unequal-mass triple systems have been found in previous theoretical or numerical studies. Fortunately, the *Gaia* mission [42] has produced the high-precision measurements of positions and motions of nearly 1.7 billion stars, which provide a major resource for studying non-hierarchical periodic triple systems. This suggests that our newly found stable *non-hierarchical* periodic orbits are likely to be observed in the near future.

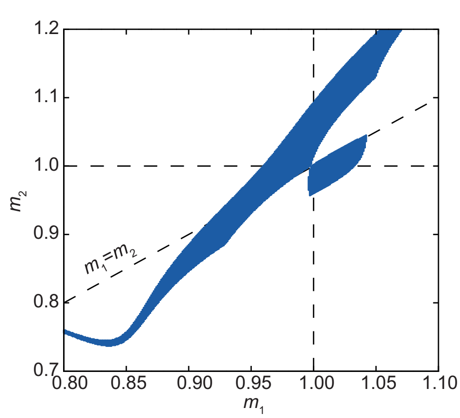

Figure 3 (Color online) The stability region of periodic orbits with $m_3 = 1$ in the m_1-m_2 plane. Shadowing domain: stable periodic orbits.

In this paper, we present one family of 135445 periodic orbits for non-hierarchical triple system with unequal masses. Surprisingly, among the 135445 periodic orbits of this family, 13315 periodic orbits are linearly stable in a large mass region. Most of them have fairly different masses, which suggest that our numerically found stable periodic orbits are likely to be observed in practice. Our numerical approach also has general meanings. Although we have only considered here one family of periodic orbits corresponding to the free group element *bABabaBAba*, we have found 13315 stable orbits among the 135445 periodic orbits. We emphasize that thousands of families of periodic orbits of three-body systems with two or three equal masses have been found to date [9, 10, 32]: each of them could be used similarly as a starting point to generate thousands of stable periodic orbits of triple systems with distinctly unequal masses (but with the same free group element). Therefore, in theory, there should exist an infinite number of stable periodic orbits of non-hierarchical triple systems with distinctly unequal masses. These newly found stable periodic orbits of non-hierarchical unequal-mass triple systems have broad impact for astrophysics: they may inspire theoretical and observational studies of non-hierarchical triple systems, the formation of triple stars [1], gravitational waves patterns [43] and gravitational waves observations [44] of non-hierarchical triple systems. Note that the periodic orbits and the stability analysis we have reported here are two-dimensional. In the future it will be valuable to search for stable *three-dimensional* periodic orbits of unequal-mass triple systems.

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Supporting Information

The supporting information is available online at phys.scichina.com and http://link.springer.com/journal/11433. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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