

Example 2: a system of coupled ODEs with algebraic property at infinity

Consider a set of two coupled nonlinear differential equations⁵

$$f'''(\eta) + \theta(\eta) - f'^2 = 0, \quad (10)$$

$$\theta''(\eta) = 3\sigma f'(\eta)\theta(\eta), \quad (11)$$

subject to

$$f(0) = f'(0) = 0, \theta(0) = 1, f'(+\infty) = \theta(+\infty) = 0, \quad (12)$$

where the prime denotes differentiation with respect to the similarity variable η , σ is the Prandtl number, $f(\eta)$ and $\theta(\eta)$ relate to the velocity profile and temperature distribution of the boundary layer, respectively. Liao⁵ employed the HAM to solve this system analytically. Now we use the BVP4c 2.0 to solve it.

Under the transformation

$$\xi = 1 + \lambda\eta, F(\xi) = f'(\eta), S(\xi) = \theta(\eta), \quad (13)$$

Eqs. (10) and (11) become

$$\lambda^2 F''(\xi) + S(\xi) - F^2(\xi) = 0, \quad (14)$$

$$\lambda^2 S''(\xi) = 3\sigma F(\xi)S(\xi), \quad (15)$$

subject to

$$F(1) = 0, S(1) = 1, F(+\infty) = S(+\infty) = 0. \quad (16)$$

Since there are two ODEs in system (14)–(15) without an unknown to be determined, we have NumEQ = 2 and TypeEQ=1. This new system is defined as

```

TypeEQ = 1;
NumEQ = 2;
f[1,z_,{F_,S_},Lambda_] := 1*a^2*D[F,{z,2}] + S - F^2;
f[2,z_,{F_,S_},Lambda_] := 1*a^2*D[S,{z,2}] - 3*sigma*F*S;

```

The four boundary conditions (16) are defined as

```

NumBC = 4;
BC[1,z_,{F_,S_}] := F /. z -> 1;
BC[2,z_,{F_,S_}] := (G - 1) /. z -> 1;
BC[3,z_,{F_,S_}] := F /. z -> infinity;
BC[4,z_,{F_,S_}] := G /. z -> infinity;

```

Now let us input the solution intervals and integral intervals for computing squared residual error

```

zL[1] = 1;
zR[1] = infinity;
zL[2] = 1;
zR[2] = infinity;
zRintegral[1] = 10;
zRintegral[2] = 10;

```

The initial guesses are chosen as $F_0 = \gamma(\xi^{-2} - \xi^{-3})$, $S_0 = \xi^{-4}$ and they are input as

```

U[1, 0] = gamma*(z^(-2) - z^(-3));
U[2, 0] = z^(-4);

```

The auxiliary linear operators are $\mathcal{L}_F = \frac{\xi}{3} \frac{\partial^2}{\partial \xi^2} + \frac{\partial}{\partial \xi}$ and $\mathcal{L}_S = \frac{\xi}{5} \frac{\partial^2}{\partial \xi^2} + \frac{\partial}{\partial \xi}$, which are defined as

```

L[1, u_] := D[u, {z, 2}]*z/3 + D[u, z];
L[2, u_] := D[u, {z, 2}]*z/5 + D[u, z];

```

Without loss of generality, let us consider the case when $\sigma = 1$, $\gamma = 3$ and $\lambda = 1/3$. These physical parameters and the control parameters $c0[k]$ are defined as

```

sigma = 1;
gamma = 3;
la = 1/3;
c0[1] = -1/2;
c0[2] = -1/2;

```

Then we call the main module `BVPh`

```

BVPh[1, 20];

```

to get the 20th-order approximations. If we are not satisfied with the accuracy of the 20th-order approximation, we can use `BVPh[21, 40]`, instead of `BVPh[1, 40]`, to get 40th-order approximation or higher order approximation.

Note that `U[1, 40]` and `U[2, 40]` are the 40th-order approximations of the transformed system (14), (15) and (16). To plot the curve of the 40th-order approximations for the original problem, we first replace z with $1 + \lambda\eta$ to obtain the 40th-order approximations for $f'(\eta)$ and $g(\eta)$, then plot the curve we want. This is done in Mathematica by the following command

```

trans = {z -> 1 + la*\[Eta]};
Plot[Evaluate[{U[1, 40], U[2, 40]} /. trans],
{\[Eta], 0, 10}, PlotRange -> {{0, 10}, {0, 1}},
AxesLabel -> {"\[Eta]", ""},
PlotStyle -> {{Thin, Red}, {Dashed, Blue}}]

```

and the curve is shown in Fig. 5. Here $\text{trans}=\{z\rightarrow 1+la*\eta\}$ is the corresponding transformation, η is the symbol η in Mathematica..

The total error $\text{ErrTotal}[k]$ of the transformed system for every two order approximations is plotted in Fig. 6 by the following command

```

ListLogPlot[Table[{2 i, ErrTotal[2*i]}, {i, 1, 20}],
Joined -> True, Mesh -> All,
PlotRange -> {{2, 40}, {10^(-10), 0.01}},
AxesLabel -> {"m", "error"}]

```

Note that $\text{ErrTotal}[k]$ not only measures the accuracy of the k th-order approximations for the transformed problem, but also measures the corresponding approximations for the original problem.

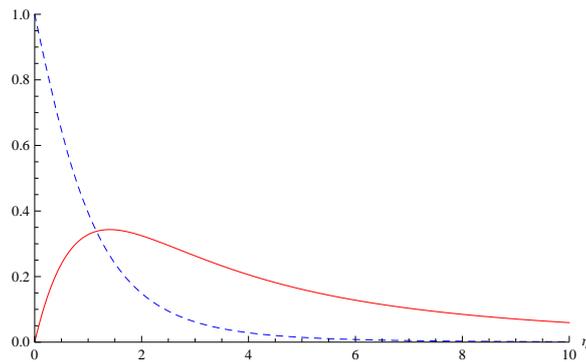


Fig. 5: The curve of $f'(\eta)$ (solid) and $\theta(\eta)$ (dashed) for Example 2.

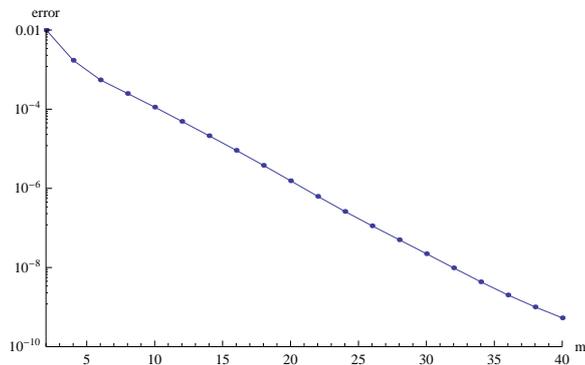


Fig. 6: Total error vs. order of approximation for Example 2.

The 40th-order approximations of $f''(0)$ and $g''(0)$ are 0.693268 and -0.769879 , respectively. Kuiken's numerical result is $f''(0) \approx 0.693212$ and $g'(0) \approx -0.769861$. To get more accurate result, we have two choices. One is to call the module `BVPh` to get higher order approximation as before, the other is to apply the Padé approximation to the current approximations. The latter is done by calling the module `hp` as follows

```
hp[Table[D[(U[1,i]/.trans),\[Eta]]/\[Eta]->0,
{i,0,40}],20,20]
hp[Table[D[(U[2,i]/.trans),\[Eta]]/\[Eta]->0,
{i,0,40}],20,20]
```

which give 0.693212 and -0.769861 , the $[20, 20]$ homotopy-Padé approximations of $f''(0)$ and $g'(0)$, respectively.

Note that we can compare the curve of $2n$ th-order approximation and the $[n, n]$ homotopy-Padé approximation in a simple and efficient way. Here we compare `U[1,40]` and the $[20, 20]$ homotopy-Padé approximations of `U[1,i]`, $i = 0 \dots 40$, in the Mathematica by the following command.

```
Plot[{U[1, 40]/.trans, hp[Table[U[1,i]/.trans,
{i, 0, 40}],20, 20]},{\[Eta],0,10},PlotRange->Full,
AxesLabel->{"\[Eta]", ""},
PlotStyle->{{Thin,Red},{Dashed,Blue}}
```

The comparison is shown in Fig. 7. From it we can see that the two are almost the same. This validate the convergence of the approximations to some extent. The above command is very efficient, because the `Plot` command in Mathematica first substitute the sample points into the expression and then applies the `hp` to a list of numerical values, rather than applies the `hp` to a list of expressions and then substitute the sample points into the resulting expression.

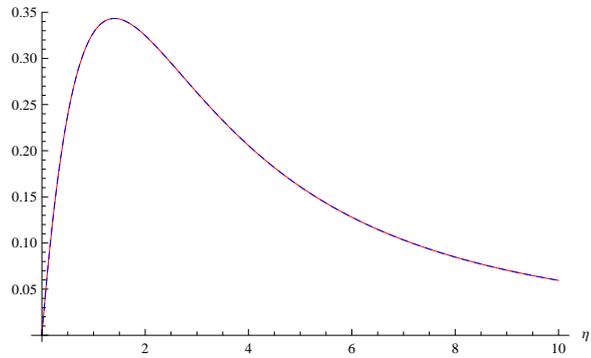


Fig. 7: The curve of 40th-order approximation of $f'(\eta)$ (solid) and $[20, 20]$ homotopy-Padé approximations of $f'(\eta)$ (dashed) for Example 2.

References

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