Example 3: a system of ODEs with an unknown parameter

Consider a system of ODEs[10]

$$U'' + (GrPr)\theta - N_r\phi + \sigma = 0, \tag{17}$$

$$\theta'' + N_b\theta'\phi' + N_t(\theta')^2 + N_b + N_t - U = 0, \tag{18}$$

$$\phi'' + \frac{N_t}{N_b}\theta'' - L_eU = 0, \tag{19}$$

subject to

$$U(-1) = U(1) = 0,\ \theta(-1) = \theta(1) = 0,\ \phi(-1) = \phi(1) = 0, \tag{20}$$

with an additional condition

$$\int_0^1 U dY = RePr, \tag{21}$$

where Gr is the Grashof number, Pr the Prandtl number, Nr the buoyancy ratio, σ the pressure parameter, Nb the Brownian motion parameter, Nt the thermophoresis parameter, Le the Lewis number, and Re the Reynolds number. All of the above parameters will be given for a special case except σ, which is to be determined from the system. Xu[10] solved this problem by the HAM.

Here we solve this problem by `BVPh 2.0`. Since there are three ODEs in system (17)–(19) with an unknown σ to be determined, we have `NumEQ = 3` and `TypeEQ=2`. The system is input as

```
TypeEQ = 2;
NumEQ = 3;
f[1,z_,{f_,g_,s_},sigma_] :=
    D[f,{z,2}]+Gr*Pr*g-Nr*s+sigma;
f[2,z_,{f_,g_,s_},sigma_] :=
    D[g,{z,2}]+Nb*D[g,z]*D[s,z]+Nt*(D[g,z])^2-f;
f[3,z_,{f_,g_,s_},sigma_] :=
    D[s,{z,2}]+Nt/Nb*D[f,{z,2}]-Le*f;
```

The seven boundary conditions, including the additional condition (21), are defined as

```
NumBC = 7;
BC[1,z_,{f_,g_,s_}] :=f/.z->-1;
BC[2,z_,{f_,g_,s_}] :=f/.z->1;
BC[3,z_,{f_,g_,s_}] :=g/.z->-1;
BC[4,z_,{f_,g_,s_}] :=g/.z->1;
BC[5,z_,{f_,g_,s_}] :=s/.z->-1;
BC[6,z_,{f_,g_,s_}] :=s/.z->1;
BC[7,z_,{f_,g_,s_}] :=Integrate[f,{z,0,1}]-Ra*Pr;
```

Now let us input the solution intervals

```
zL[1] = -1;    zR[1] = 1;
zL[2] = -1;    zR[2] = 1;
zL[3] = -1;    zR[3] = 1;
```

Since all the solution intervals are in finite intervals, we do not have to specify the integral interval to compute the squared residual error.

The initial guesses are chosen as $U_0 = \epsilon_1 - 3(-25+\epsilon_1)z^2/2 + 5(-15+2\epsilon_1)z^4/2$, $\theta_0 = \epsilon_2(1-z^2)$ and $\phi_0 = \epsilon_3(1-z^2)$, where ϵ_1, ϵ_2 and ϵ_3 are constants to be optimized. They are input as

```
U[1,0]=eps1-3/2*(-25+4eps1)z^2+5/2*(-15+2eps1)*z^4;
U[2,0]=eps2*(1-z^2);
U[3,0]=eps3*(1-z^2);
```

The auxiliary linear operators are chosen as $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_3 = \frac{\partial^2}{\partial Y^2}$. They are defined as

```
L[1, u_] := D[u, {z, 2}];
L[2, u_] := D[u, {z, 2}];
L[3, u_] := D[u, {z, 2}];
```

Note that we use the delayed assignment `SetDelayed(:=)` to define these linear operators.

Without loss of generality, let us consider the case when $Nr = 3/20$, $Nt = Nb = 1/20$, $Le = 10$, $Gr = 5$, $Pr = 1$ and $Re = 5$. These physical parameters are input as

```
Nr = 3/20;     Nt = 1/20;
Nb = 1/20;     Le = 10;
Gr = 5;        Pr = 1;
Ra = 5;
```

At this time, we have input all the data for this problem, except the convergence-control parameters `c0[k]`, `eps1`, `eps2` and `eps3`. We minimize the squared residual error of the 3th-order approximations to get the optimal values by the module `GetOptiVar` as follows

```
c0[1] = c0[2] = c0[3] = h;
GetOptiVar[3, {}, {eps1, eps2, eps3, h}];
```

Note that we put constraints `c0[1]`=`c0[2]`=`c0[3]` on `c0[1]`, `c0[2]` and `c0[3]` to simplify the computation. There is no constraint on `eps1`, `eps2` and `eps3`.

After some computation, we get optimal values for all the convergence-control parameters c0[1] = c0[2] = c0[3] ≈ -0.769452, eps1 ≈ 7.56408, eps2 ≈ -2.58887 and eps3 ≈ -30.0044. Now we can use

```
BVPh[1,10]
```

to get the 10th-order approximation.

If we are not satisfied with the accuracy of the 10th-order approximation, we can use BVPh[11,20] to get 20th-order approximation or higher order approximation. The 20th-order approximations of U, θ and ϕ are stored in U[1,20], U[2,20] and U[3,20], the 20th-order approximation of σ is stored in Lambda[19], while the corresponding squared residual error is ErrTotal[20]. Lambda[19] is about 18.272555944, which is the same with Xu's result.[10] The 20th-order approximations are plotted in Fig. 8. The total error of the system for every two order of approximations are plotted in Fig. 9.

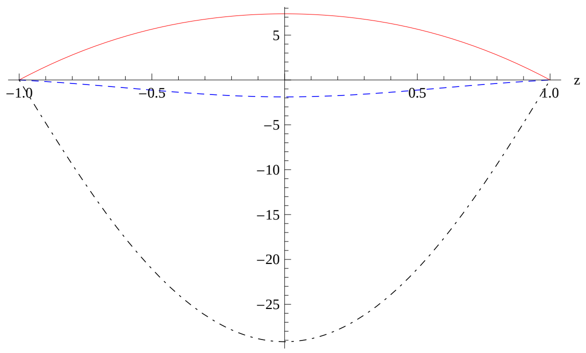

Fig. 8: The curve of U (solid), θ (dashed) and $\phi(z)$ (dot dashed) for Example 3.

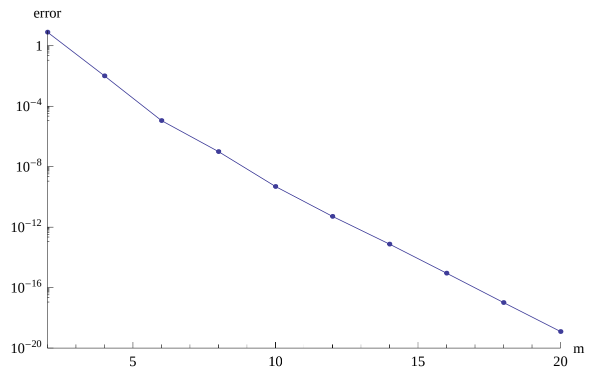

Fig. 9: Total error vs. order of approximation for Example 3.

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