Example 5: iterative solutions of the Gelfand equation

If the problem is defined in a finite interval, the BVPh 1.0 can solve it using an iterative method. The BVPh 2.0 has inherited this feature. However, there are some minor differences in the input.

Consider the Gelfand equation[12–14]

$$u'' + (K-1)\frac{u'}{z} + \lambda e^u = 0, \ u'(0) = u(1) = 0, \tag{30}$$

where the prime denotes the differentiation with respect to z, $K \geq 1$ is a constant, $u(z)$ and λ denote eigenfunction and eigenvalue, respectively. Following Liao[6] , an additional boundary condition

$$u(0) = A \tag{31}$$

is added to distinguish different eigenfunctions.

To solve this problem by `BVPh 2.0`, we have to input the differential equations, boundary conditions and initial guesses. Since the problem is a single ODE with an unknown λ to be determined, we set `NumEQ = 1` and `TypeEQ=2`. The differential equation can be coded as follows

```
TypeEQ = 2;
NumEQ = 1;
f[1,z_,{u_},lambda_] :=
  D[u,{z,2}] +(K-1)*D[u,z]/z+lambda*Exp[u];
```

The three boundary conditions, including the additional condition (31), are defined as

```
NumBC = 3;
BC[1, z_, {u_}] := (u-A)/. z -> 0;
BC[2, z_, {u_}] := D[u,z]/. z -> 0;
BC[3, z_, {u_}] := u /. z -> 1;
```

Now let us input the solution intervals

```
zL[1] = 0;    zR[1] = 1;
```

Since the solution interval is finite, we do not have to specify the integral interval to compute the squared residual error.

The initial guess is chosen as $U_0 = \frac{A}{2}[1 + \cos(\pi z)]$, which is input as

```
U[1,0] = A/2*(1 + Cos[Pi*z]);
```

The auxiliary linear operator is chosen as $\mathcal{L} = \frac{\partial^2}{\partial z^2} + \left(\frac{\pi}{a}\right)^2$, which is defined as

```
    L[1,f_] :=  D[f,{z,2}]+Pi^2*f;
```

Note that we use the delayed assignment `SetDelayed(:=)` to define the linear operator.

Without loss of generality, let us consider the case when $A = 1$ and $K = 2$. The physical parameters are input as

```
    K = 2;    A = 1;
```

Because we want to approximate the right-hand sides using the hybrid-base function and use an iterative approach to get the approximations, the control parameters in BVPh 2.0 are modified to

```
    TypeL      = 2;
    HYBRID     = 1; (* hybrid-base functions *)
    TypeBase   = 2; (* even Fourier series   *)
    ApproxQ    = 1;
    Ntruncated = 30;
```

Here `TypeL=2`, `HYBRID=1` and `ApproxQ=1` together mean that the right-hand side term of all high-order deformation equations is approximated by the hybrid-base approximations. `TypeBase=2` means the even Fourier series is used (`TypeBase=1` also applies to this problem). `Ntruncated=20` means $N_t = 30$.

At this time, we have input all the data for this problem, except the convergence-control parameter `c0[1]`. To get optimal `c0[1]`, we minimize the squared residual error of the 6th-order approximations. This is done in BVPh 2.0 by calling the function `GetOptiVar`

```
    GetOptiVar[6, {}, {c0[1]}];
```

After some computation, we get the optimal value for the convergence-control parameter `c0[1]`$= -0.522418$. Now we can use the 3rd-order iteration HAM approach

```
    iter[1,6,3]
```

to get the desired approximation. Here 6 means the iteration times. After about 40 seconds, the 6th iteration gives the eigenvalue 1.90921, which is the same with Liao's result.[6] The kth iteration approximations of u and λ are stored in `V[1,k]`, and `LAMBDA[k]`, while the corresponding squared residual error is stored in `ERRTotal[k]`. The 6th iteration approximation is plotted in Fig. 12 by

```
Plot[V[1,6],{z,0,1},AxesLabel->{"z", "u(z)"}]
```

The total error for each iteration is plotted in Fig. 13 by

```
ListLogPlot[Table[{i, ERRTotal[i]}, {i, 1, 6}],
PlotRange->{{1,6},{10^-10,0.01}},Joined->True, Mesh->All,
AxesLabel->{"m","error"}]
```
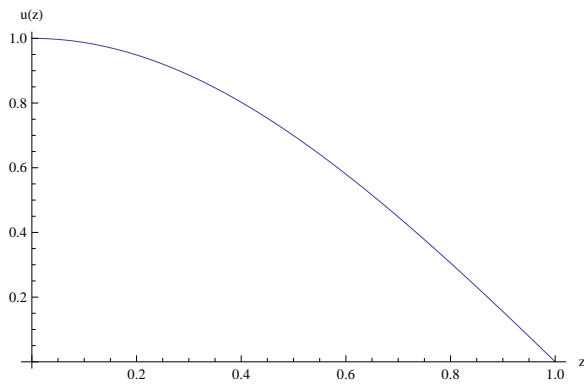
Fig. 12: The curve of the eigenfunction $u(z)$ corresponding to the eigenvalue $\lambda = 1.90921$ when $A = 1$ and $K = 2$ for Example 5.
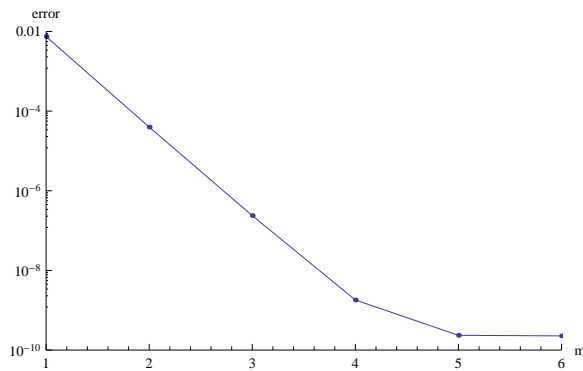
Fig. 13: Total error for each iteration vs. iteration times m for Example 5.

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