Basic Ideas and Brief History of the Homotopy Analysis Method

1 Introduction

Nonlinear equations are much more difficult to solve than linear ones, especially by means of analytic methods. In general, there are two standards for a satisfactory analytic method of nonlinear equations:

(a) It can *always* provide analytic approximations *efficiently*.

(b) It can ensure that analytic approximations are *accurate* enough for *all* physical parameters.

Using above two standards as criterion, let us compare different analytic techniques for nonlinear problems.

Perturbation techniques [11, 15, 30, 37, 38, 43] are widely applied in science and engineering. Perturbation techniques are based on small/large physical parameters (called perturbation quantities) in governing equations or initial/boundary conditions. In general, perturbation approximations are expressed in series of perturbation quantities, and a nonlinear equation is replaced by an infinite number of linear (sometimes even nonlinear) sub-problems, which are completely determined by the original governing equation and especially by the place where perturbation quantities appear. Perturbation methods are simple, and easy to understand. Especially, based on small physical parameters, perturbation approximations often have clear physical meanings. Unfortunately, *not* every nonlinear problem has such kind of perturbation quantity. Besides, even if there exists such a small physical parameter, the sub-problem might have no solutions, or might be so complicated that only a few sub-problems can be solved. Thus, it is *not* guaranteed that one can *always* gain perturbation approximations for any a given nonlinear problem. More importantly, it is well-known that most perturbation approximations are valid only for small physical parameters. So, it is *not* guaranteed that a perturbation result is valid in the whole region of *all* physical parameters. Thus, perturbation techniques do not satisfy not only the standard (a) but also the standard (b) mentioned above.

To overcome the restrictions of perturbation techniques, some traditional non-perturbation methods are developed, such as Lyapunov's artificial small parameter method [31], the δ-expansion method [9, 14], Adomian decomposition method [4–7, 10, 41], and so on. In principle, all of these methods are based on a so-called artificial parameter, and approximation solutions are expanded into series of such kind of artificial parameter. The artificial small parameter is often used in such a way that one can always gain approximation solutions for any a given nonlinear equation. Compared with perturbation techniques, this is indeed a great progress. However, in theory, one can put the artificial small parameter in many different ways, but unfortunately there are no theories to guide us how to put it in a better place so as to gain a better approximation. For example, Adomian decomposition method simply uses the linear operator d^k/dx^k in most cases, where k is the highest order of derivative of governing equations, and therefore it is rather easy to gain solutions of the corresponding sub-problems by means of integrating k times with respect to x. However, such simple linear operator gives approximations in power-series, which unfortunately has often a finite radius of convergence. Thus, Adomian decomposition method can not ensure the convergence of its approximation series. Generally speaking, all traditional non-perturbation methods, such as Lyapunov's artificial small parameter method [31], the δ-expansion method [9, 14] and Adomian decomposition method [4–7, 10, 41], can *not* guarantee the convergence of approximation series. So, these traditional non-perturbation methods satisfy only the standard (a) but *not* the standard (b) mentioned before.