3 The normal HAM

Unfortunately, Liao [18,20] found that the early HAM mentioned above can not always guarantee the convergence of approximation series of nonlinear equations in general. To overcome this restriction, Liao [18] in 1997 introduced such a non-zero auxiliary parameter c_0 to construct a *two-parameter* family of equations*, i.e. the zeroth-order deformation equation

$$(1-q)\mathcal{L}\left[\phi(x;q)-u_0(x)\right] = c_0\, q\, \mathcal{N}[\phi(x;q)], \quad x\in\Omega,\ q\in[0,1]. \tag{6}$$

The corresponding high-order deformation equation reads

$$\mathcal{L}\left[u_n(x)-\chi_n\, u_{n-1}(x)\right] = c_0\, \delta_{n-1}(x), \tag{7}$$

where $\delta_k(x)$ is defined by (5). In this way, the homotopy-series solution (3) is not only dependent upon the physical variable x but also the auxiliary parameter c_0. Mathematically, it was found [18–20] that the auxiliary parameter c_0 can adjust and control the convergence region and rate of homotopy-series solutions, although c_0 has no physical meanings at all. For detailed mathematical proofs, please refer to Chapter 5 of [29]. In essence, the use of the auxiliary parameter c_0 introduces us one more "artificial" degree of freedom, which greatly improves the early HAM: it is the auxiliary parameter c_0 which provides us a convenient way to guarantee the convergence of homotopy-series solution. For example, Liang & Jeffrey [16] illustrated that, when analytic approximations given by the other analytic method is divergent in the whole domain, one can gain convergent series solution simply by choosing a proper auxiliary parameter c_0. This is the reason why we call c_0 the *convergence-control parameter*.

The use of the convergence-control parameter c_0 is indeed a great progress in the frame of the HAM. It seems that more "artificial" degrees of freedom imply larger possibility to gain better approximations by means of the homotopy analysis method. Thus, Liao [19] in 1999 further introduced more "artificial" degrees of freedom by using the zeroth-order deformation equation in a more general form:

$$[1-\alpha(q)]\mathcal{L}\left[\phi(x;q)-u_0(x)\right] = c_0\, \beta(q)\, \mathcal{N}[\phi(x;q)], \quad x\in\Omega,\ q\in[0,1], \tag{8}$$

where $\alpha(q)$ and $\beta(q)$ are the so-called *deformation functions*† satisfying

$$\alpha(0)=\beta(0)=0, \alpha(1)=\beta(1)=1, \tag{9}$$

whose Taylor series

$$\alpha(q)=\sum_{m=1}^{+\infty}\alpha_m\, q^m, \quad \beta(q)=\sum_{m=1}^{+\infty}\beta_m\, q^m, \tag{10}$$

*Liao [18] originally used the symbol $\hbar$ to denote the non-zero auxiliary parameter. But, $\hbar$ is well-known as Planck's constant in quantum mechanics. To avoid misunderstanding, $\hbar$ is replaced by the symbol c_0 in the book [29], which denotes the "basic" convergence-control parameter.

†$\alpha(q)$ and $\beta(q)$ were called "approaching function" in some early articles about the homotopy-analysis method. In the book [29], they are defined as "deformation function", which better reveals its relationship with the zeroth-order deformation equations

are convergent for $|q| \leq 1$. The corresponding high-order deformation equation reads

$$\mathcal{L}\left[u_m(x) - \sum_{k=1}^{m-1} \alpha_k \, u_{m-k}(x)\right] = c_0 \sum_{k=1}^{m} \beta_k \, \delta_{m-k}(x), \tag{11}$$

where $\delta_k(x)$ is defined by (5).

In fact, the zeroth-order deformation equation (8) can be further generalized, as shown by Liao [20,21,24]. Obviously, there are an infinite number of the deformation functions as defined above. Thus, the approximation series given by the HAM can contain so many "artificial" degrees of freedom that they provide us great possibility to guarantee the convergence of homotopy-series solution. Note that $u_n(x)$ is always governed by the same auxiliary linear operator $\mathcal{L}$, and we have great freedom to choose $\mathcal{L}$ in such a way that $u_n(x)$ is easy to obtain. More importantly, for given auxiliary linear operator $\mathcal{L}$ and initial guess, we can always gain convergent homotopy-series solution by choosing proper convergence-control parameter c_0 and proper deformation functions $\alpha(q)$ and $\beta(q)$. Inversely, the guarantee of the convergence of homotopy-series solutions also provides us freedom to choose the auxiliary linear operator $\mathcal{L}$ and initial guess. It is due to such kind of guarantee in the frame of the HAM that a nonlinear ODE with variable coefficients can be transferred into a sequence of linear ODEs with constant coefficients [26], that a nonlinear PDE can be transferred into an infinite number of linear ODEs [22,25], that several coupled nonlinear ODEs can be transferred into an infinite number of linear decoupled ODEs [45], and that even a 2nd-order nonlinear PDE can be replaced by an infinite number of 4th-order linear PDEs [23]. In fact, it is such kind of guarantee for convergence of series solutions, together with the extremely large freedom in choice of the auxiliary linear operators, that greatly simplifies finding convergent series of nonlinear equations in the frame of the HAM, as illustrated in above-mentioned articles [22,23,25,26,45]. On the other hand, without such kind of guarantee of convergence, we have in practice no *true* freedom to choose the auxiliary linear operator $\mathcal{L}$, because the freedom to get a divergent series solution has no meanings at all! For example, Liang & Jeffrey [16] pointed out that the series solution given by means of the so-called "homotopy perturbation method¨ [12] is divergent at all points except the initial guess, and thus has completely no scientific meanings. So, unlike perturbation techniques and the traditional non-perturbation methods mentioned above, the HAM satisfies both the standard (a) and (b).

How to find a proper convergence-control parameter c_0 so as to gain a convergent series solution? A straight-forward way to check the convergence of a homotopy-series solution is to substitute it into original governing equations and boundary/initial conditions and then to check the corresponding squared residual integrated in the whole region. However, when the approximations contain unknown convergence-control parameters and/or other unknown physical parameters, it is time-consuming to calculate the squared residual at high-order of approximations. To avoid the time-consuming computation, Liao [18–20] suggested to investigate the convergence of some special quantities which often have important physical meanings. For example, one can consider the convergence of $u'(0)$ and $u''(0)$ of a nonlinear differential equation

$\mathcal{N}[u(x)] = 0$. It is found by Liao [18–20] that there often exists such an effective-region $\mathbf{R}_c$ that any $c_0 \in \mathbf{R}_c$ gives a convergent series solution of such kind of quantities. Besides, such kind of effective-region can be found, although approximately, by plotting the curves of these unknown quantities versus c_0. For example, for a nonlinear differential equation $\mathcal{N}[u(x)] = 0$, one may approximately determine $\mathbf{R}_c$ by plotting curves $u'(0) \sim c_0$, $u''(0) \sim c_0$ and so on. These curves are called "c_0-curves" or "curves for convergence-control parameter"[‡], which have been successfully applied to solve many nonlinear problems [20].

[‡]The c_0-curve was originally called the $\hbar$-curve, and $\mathbf{R}_c$ was originally denoted by $\mathbf{R}_\hbar$.