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Ultra-chaos of a mobile robot: A higher disorder than normal-chaos

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ABSTRACT

The trajectory of a chaotic dynamical system is sensitive to the initial condition, while its statistics normally do not have sensitive dependence on small disturbances. Such a kind of chaos is called normal-chaos. Recently, a new concept named ultra-chaos has been proposed for the first time. Unlike normal-chaos, the small disturbances lead to large deviation even in statistics for ultra-chaos. However, few ultra-chaos have been reported up to now. In this work, we investigate the influences of small disturbances on the chaotic navigation system and the motion of mobile robot, respectively. It is illustrated that the chaotic navigation system of the mobile robot belongs to normal-chaos. However, the motion of chaotic mobile robot belongs to ultra-chaos. Tiny disturbances would cause huge deviation in the motion of chaotic mobile robot, even in the statistical meaning. Thus, the ultra-chaos is indeed a higher-level disorder compared with the normal-chaos and it widely exists in practice. Hopefully, the new concept, i.e., ultra-chaos, could deepen our understandings of chaos and open a new door to explore chaos theory.

1. Introduction

It is well known that nonlinear dynamical systems widely exist in science and engineering [1–3]. In 1890, Poincaré [4] first discovered that a chaotic dynamical system has sensitivity dependence on the initial condition. Lorenz [5] rediscovered this property in 1963 with a more popular name, i.e. "butterfly-effect". Therefore, for a chaotic dynamical system, a small variation of the initial condition could lead to a large deviation in numerical simulation of trajectories after a long enough time [6–8]. Besides, it was Lorenz [9] that further discovered that the chaotic system also has sensitivity dependence on numerical simulations: computer-generated simulations of a chaotic system obtained by the traditional algorithm in single or double precision are sensitive to both the initial conditions and the numerical algorithms. Significantly, for numerical algorithms, the numerical noises including the truncation errors and round-off errors are *unavoidable*. Naturally, this kind of unreliability of chaotic trajectories leads to the heated debate about whether the numerical simulation of a chaotic dynamical system is credible [10]. Some even gave the pessimistic remark that the numerical convergence of chaotic system cannot be guaranteed *forever* [11]. In addition, it was reported recently by Chandramoorthy and Wang that shadowing solution can be almost nonphysical, and this invalidates the argument that small perturbation only has slight influence on its statistical behavior in a chaotic system [12]. Chaotic dynamics [13–17] is often regarded as one of the three greatest theories in physics in the 20th century, comparable to quantum mechanics and Einstein's relativity theory.

To overcome the limitation of traditional algorithm in single/double precision and obtain a *reliable* numerical simulation of chaotic system, in 2009, Liao [18] suggested a numerical method, called the "clean numerical simulation" (CNS). In the framework of the CNS [18–20], the round-off error can be decreased to a *required* tiny level, through implementing a large *enough* number of significant digits in multiple precision to represent all parameters and variables [21]. Besides, the temporal and spatial truncation errors are decreased to a sufficiently small level by implementing the Taylor expansion with a high *enough* order. Furthermore, it is necessary to perform another simulation with even smaller numerical noise for determining the T_c, called "critical predictable time", so that the numerical noise can be negligible within $t \in [0, T_c]$. Therefore, the CNS can provide us with a reliable simulation of chaos within a long enough interval of time. The CNS has been successfully used to deal with lots of chaotic dynamical systems such as the Lorenz system [18], three-body system [22], and some spatio-temporal chaos [19,20,23], which confirms the validity, novelty, and great potential for using the CNS to deal with chaotic dynamical system.

To study the influence of numerical noises on chaos, a numerical simulation obtained by the CNS can be applied as a benchmark solution. By means of the CNS, it is found that tiny disturbances might

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cause huge deviation of some temporal and spatiotemporal chaos even in *statistics* [19,20]. Realizing the significance of such kind of chaos, we proposed a new concept "ultra-chaos" [24]. For a normal chaos, although its trajectories are rather sensitive to tiny disturbances, its statistics remain the same. However, different from the normal-chaos, the statistics of an ultra-chaos such as the probability density function (PDF) are extremely sensitive to small noises/disturbances. Note that, the hyper-chaos is typically defined through the Lyapunov exponents, as it is a chaos with at least two positive Lyapunov exponents [25]. However, the ultra-chaos is defined through the statistical instability. The statistics of an ultra-chaos are rather sensitive to small disturbances. It has been shown that the hyper-chaotic Rössler system with two positive Lyapunov exponents belongs to normal chaos rather than ultra-chaos, because its statistics are insensitive to the tiny disturbances [24]. Thus, ultra-chaos contains something completely new. It is a pity that quite few ultra-chaos have been reported.

In recent years, autonomous mobile robot has become a topic of great interest. There are many applications of mobile robots in civil, industrial, and military activities [26–28]. This happens because many interesting applications in dangerous or difficult tasks for human can be replaced by robots. Especially, the autonomous robots with a chaotic navigation system have been explored a lot for various purposes such as exploration missions [29–31], patrol missions [32,33], surveillance [34–36], fire fighting [37], etc. Chaotic signals are used in these applications, and they can be generated through a chaotic circuit to control the kinetic state of mobile robots [38]. Note that, chaotic dynamical systems provide the necessary framework to realize the previously mentioned tasks because of the sensitivity to initial conditions, their topological transitivity, and their functionality [31]. The chaotic mobile robot is capable of analyzing the whole connected workplace, because of the property of topological transitivity [29]. If the task of a mobile robot is only to explore a terrain efficiently, the classical navigation system like a systematic scan with the parallel straight strategy might require less time compared with the chaotic navigation system [39]. However, this kind of classical navigation system could be easily understood by an intruder in the case of a patrolling mission. Therefore, for intrusion patrolling, the trajectories of the mobile robot need to be much difficult for the intruder to predict. This goal can be achieved by the chaotic navigation systems, which can ensure the chaotic dynamical motion of the mobile robot [35,39].

Many dynamical chaotic systems can be applied as the navigation system to generate the chaotic motions for the mobile robot such as the Lorenz system [5], the hyper-chaotic Rössler chaotic system [40], and the Vaidyanathan system [41], which belong to normal-chaos [24]. (The normal-chaos will be discussed in detail in Section 3.1.) Without loss of generality, we take the 3D novel Vaidyanathan jerk chaotic system [38,41] and the hyper-chaotic Rössler system [24,40] as two examples to study the influences of disturbance on the navigation system and the motion of the mobile robot, respectively. In this work, the chaotic mobile robot navigated by the 3D novel Vaidyanathan jerk chaotic system is the same case as that illustrated in Ref. [38]. The specific hyper-chaotic Rössler system is the same as that described in Refs. [24,40].

In the present paper, using the CNS algorithm, we successfully gain the reliable numerical simulation of the chaotic mobile robot. Using the numerical simulation obtained by the CNS without disturbances as a benchmark solution, we study the influences of tiny disturbances on the chaotic navigation system and the motion of mobile robot, respectively. It is illustrated that the chaotic navigation system of the mobile robot belongs to normal-chaos. For normal-chaos, although small disturbances can lead to great deviations in trajectories, their statistics are insensitive to disturbances. However, it is found that the motion of chaotic mobile robot belongs to ultra-chaos, which is a higher disorder compared with the normal-chaos. For ultra-chaos, tiny disturbances can lead to huge deviation, even in statistics. In general, the CNS is an effective tool to gain reliable simulations and further study the influence of tiny disturbances on chaotic systems. The ultra-chaos as a new concept might deepen our understandings of chaos and open a new door to explore chaos theory.

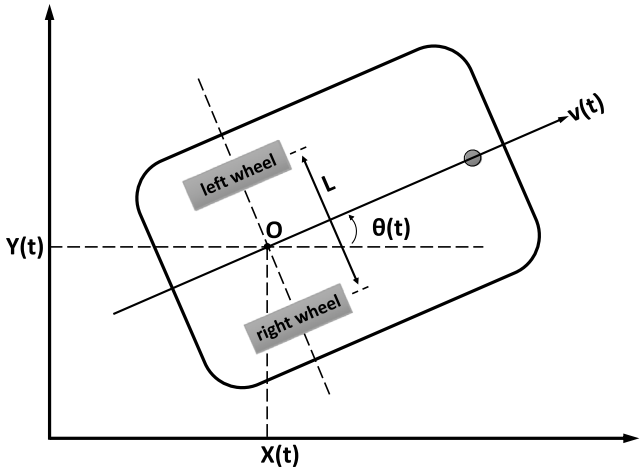

Fig. 1. Kinematic model of mobile robot.

The remainder of this manuscript is organized as follows. In Section 2, we briefly illustrate the model of the chaotic mobile robot and the CNS algorithm. The influences of disturbances on the navigation system and the motion of chaotic mobile robot are described in Section 3, respectively. Finally, the concluding remarks are illustrated in Section 4.

2. Mathematical model and CNS algorithm

In this section, the mathematical model of the chaotic mobile robot is illustrated. Besides, the CNS algorithm to gain reliable/convergent numerical simulation of the chaotic mobile robot is briefly illustrated.

The description of the two-wheeled mobile robot is shown in Fig. 1. The kinematic state equations of the differential drive mobile robot are written as follows [38]

$$\dot{X}(t) = v(t)\,cos[\theta(t)], \tag{1}$$

$$\dot{Y}(t) = v(t)sin[\theta(t)], \tag{2}$$

$$\dot{\theta}(t) = \omega(t), \tag{3}$$

where $v(t) = \frac{1}{2}[v_l(t) + v_r(t)]$ is the linear velocity of the mobile robot, v_l is the linear velocity of left wheel, v_r is the linear velocity of right wheel, $\omega(t) = \frac{1}{L}[v_r(t) - v_l(t)]$ is angular velocity, L is the distance between the two wheels, and $\theta(t)$ is the angle between the direction of the linear velocity and the horizontal direction. In this work, we take the 3D novel Vaidyanathan jerk chaotic system [38,41] and the hyper-chaotic Rössler system [24,40] as two examples of navigation system to study the influences of disturbance on the navigation system and the motion of the mobile robot, respectively. The navigation system controls the linear velocities of the left wheel $v_l(t)$ and right wheel $v_r(t)$ of the differential drive mobile robot, and the angular velocity is determined as $\omega(t) = \frac{1}{L}[v_r(t) - v_l(t)]$ [38].

2.1. Mobile robot controlled by the Vaidyanathan chaotic system

Here, we describe the chaotic mobile robot with the Vaidyanathan chaotic system controller. The motion of the mobile robot navigated by the 3D novel Vaidyanathan jerk chaotic system writes as

$$\dot{x}(t) = y(t), \tag{4}$$

$$\dot{y}(t) = z(t), \tag{5}$$

$$\dot{z}(t) = x(t) - a\,\{sinh[x(t)] + sinh[y(t)]\} - b\,z(t), \tag{6}$$

$$\dot{X}(t) = \frac{1}{2}\,[y(t) + z(t)]\,cos[\theta(t)], \tag{7}$$

$$\dot{Y}(t) = \frac{1}{2}\,[y(t) + z(t)]\,sin[\theta(t)], \tag{8}$$

$$\dot{\theta}(t) = \frac{y(t) - z(t)}{L}, \tag{9}$$

where, y is the linear velocity of right wheel, and z is the linear velocity of left wheel. Eqs. (4)–(6) present the 3D novel Vaidyanathan jerk chaotic system, which is the navigation system of the mobile robot. We take the same values of parameters a, b and L as in [38]:

$$a = 0.4, \quad b = 0.67, \quad L = 0.05, \tag{10}$$

and the initial condition

$$(x_0, y_0, z_0) = (0.8, 1.2, 0.5) + \sigma\,(0, 0, 1), \quad (X_0, Y_0, \theta_0) = (0, 0, 0), \tag{11}$$

where σ denotes a small disturbance in the initial condition. In this paper, without loss of generality, let us consider the three cases: $\sigma = 0$, $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$, respectively. When $\sigma = 0$, there is no disturbance, and the initial condition (11) is the same as that in [38].

The Lyapunov exponent of a dynamical system is a measure of exponential divergence, which characterizes the separation rate of close trajectories [42]. There are three Lyapunov exponents (L_1, L_2, L_3) in the 3D novel Vaidyanathan jerk system. Computed following [42], the three Lyapunov exponents are about

$$L_1 = 0.09, L_2 = 0, \text{and} \quad L_3 = -0.76. \tag{12}$$

The maximum Lyapunov exponent L_1 is larger than zero, which means that this Vaidyanathan jerk system has a chaotic behavior. As the sum of the Lyapunov exponents $(L_1 + L_2 + L_3 = -0.67 < 0)$ is negative, it implies that the 3D novel Vaidyanathan jerk system is dissipative [28]. The Lyapunov dimension of a chaotic system is also called the Kaplan–Yorke dimension, which can be used to estimate the Hausdorff dimension of the considered dynamical system [43,44]. The Lyapunov dimension D_{KY} can be calculated by the Kaplan–Yorke formula as

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.12. \tag{13}$$

It indicates that the Vaidyanathan jerk system is not Hamiltonian, in that case D_{KY} would be the integer equal to the phase space dimension [28]. For more dynamic characteristics of this 3D novel Vaidynathan jerk chaotic system, please refer to Refs. [38,41].

The CNS is used to gain the reliable/convergent numerical simulation of the chaotic mobile robot in a long interval of time as $t \in [0, 10000]$. To study the influences of tiny disturbances in the initial condition on the navigation system and the motion of chaotic mobile robot, we compare the numerical simulations of the chaotic mobile robot starting from the initial condition without disturbance $\sigma = 0$ and with tiny disturbances $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$.

It is emphasized here that, for each case of σ, we can always obtain a reliable/convergent numerical simulation with $t \in [0, 10000]$, by the parallel algorithm of the CNS applying the 175th-order Taylor expansion and the time step $\Delta t = 0.01$. Besides, multiple precision is used as all the data are represented in 420 significant digits. The reliability of this numerical result is guaranteed through comparing it with another result given by the CNS containing even smaller numerical noises. Another CNS result is obtained by the 180th-order Taylor expansion and the time step $\Delta t = 0.01$. All the data are represented in 440 significant digits. For the details about the CNS, please refer to [18–20]. Note that, for each σ, the numerical noises of the CNS solutions are negligible in $t \in [0, 10000]$ and thus our investigation of disturbances on the chaotic mobile robot with the Vaidyanathan jerk chaotic controller is based on the reliable chaotic simulations.

2.2. Mobile robot controlled by the hyper-chaotic Rössler system

The well-known hyper-chaotic Rössler system [24,40] is taken as another example of the navigation system to generate the chaotic motion of mobile robot. The governing equations of this chaotic mobile robot are given by,

$$\dot{x} = -\,y - z, \tag{14}$$

$$\dot{y} = x + a\,y + w, \tag{15}$$

$$\dot{z} = b + x\,z, \tag{16}$$

$$\dot{w} = -\,c\,z + d\,w \tag{17}$$

$$\dot{X}(t) = \frac{1}{2}\,[x(t) + y(t)]\,cos[\theta(t)], \tag{18}$$

$$\dot{Y}(t) = \frac{1}{2}\,[x(t) + y(t)]\,sin[\theta(t)], \tag{19}$$

$$\dot{\theta}(t) = \frac{x(t) - y(t)}{L}, \tag{20}$$

where x is the linear velocity of right wheel, and y is the linear velocity of left wheel. Eqs. (14)–(16) present the hyper-chaotic Rössler chaotic system, which is the navigation system of the mobile robot. We take the same values of parameters a, b, c, and d as in [24,40]:

$$a = 0.25, \quad b = 3, \quad c = 0.5, \quad d = 0.05, \tag{21}$$

and the initial condition

$$(x_0, y_0, z_0, \omega_0) = (-20, 0 + \sigma, 0, 15), \quad (X_0, Y_0, \theta_0) = (0, 0, 0), \tag{22}$$

where σ denotes a small disturbance in the initial condition. Without loss of generality, let us consider the three cases: $\sigma = 0$, $\sigma = 10^{-14}$, and $\sigma = 10^{-8}$, respectively. When $\sigma = 0$, there is no disturbance, and the initial condition (22) is the same as that in [24,40]. The Lyapunov exponents (L_1, L_2, L_3, L_4) of the hyper-chaotic Rössler system are about

$$L_1 = 0.11, L_2 = 0.02, L_3 = 0, \text{and} \quad L_4 = -25.3. \tag{23}$$

As the sum of the Lyapunov exponents $(L_1 + L_2 + L_3 + L_4 = -25.17 < 0)$ is negative, the hyper-chaotic Rössler system is dissipative. The Lyapunov dimension or the Kaplan–Yorke dimension D_{KY} is about

$$D_{KY} = 3 + \frac{L_1 + L_2 + L_3}{|L_4|} = 3.005, \tag{24}$$

which is a fractional dimension. For more dynamic properties of the hyper-chaotic Rössler chaotic system, please refer to the Refs. [40,45].

For each case of σ, we can always obtain a reliable numerical simulation with $t \in [0, 10000]$, by the parallel algorithm of the CNS applying the 200th-order Taylor expansion and the time step $\Delta t = 0.001$. Besides, multiple precision is used as all the data are represented in 500 significant digits. The reliability of this numerical result is guaranteed through comparing it with another result given by the CNS containing even smaller numerical noises. Another CNS result is obtained by the 202th-order Taylor expansion and the time step $\Delta t = 0.001$ with all the data represented in 510 significant digits. Thus, for each σ, our study of the influences of disturbances on the chaotic mobile robot with the Rössler chaotic controller is based on the reliable chaotic simulations in $t \in [0, 10000]$.

3. Influence of tiny disturbances on chaotic results

In this section, by means of the CNS, we investigate the influences of tiny disturbances in the initial condition on the chaotic navigation system and the motions of chaotic mobile robot, respectively.

3.1. Influence of tiny disturbances on navigation system

First, we investigate the influence of tiny disturbances on the navigation system of the chaotic mobile robot. The 3D novel Vaidyanathan jerk chaotic system and the hyper-chaotic Rössler system are used as the navigation systems, respectively.

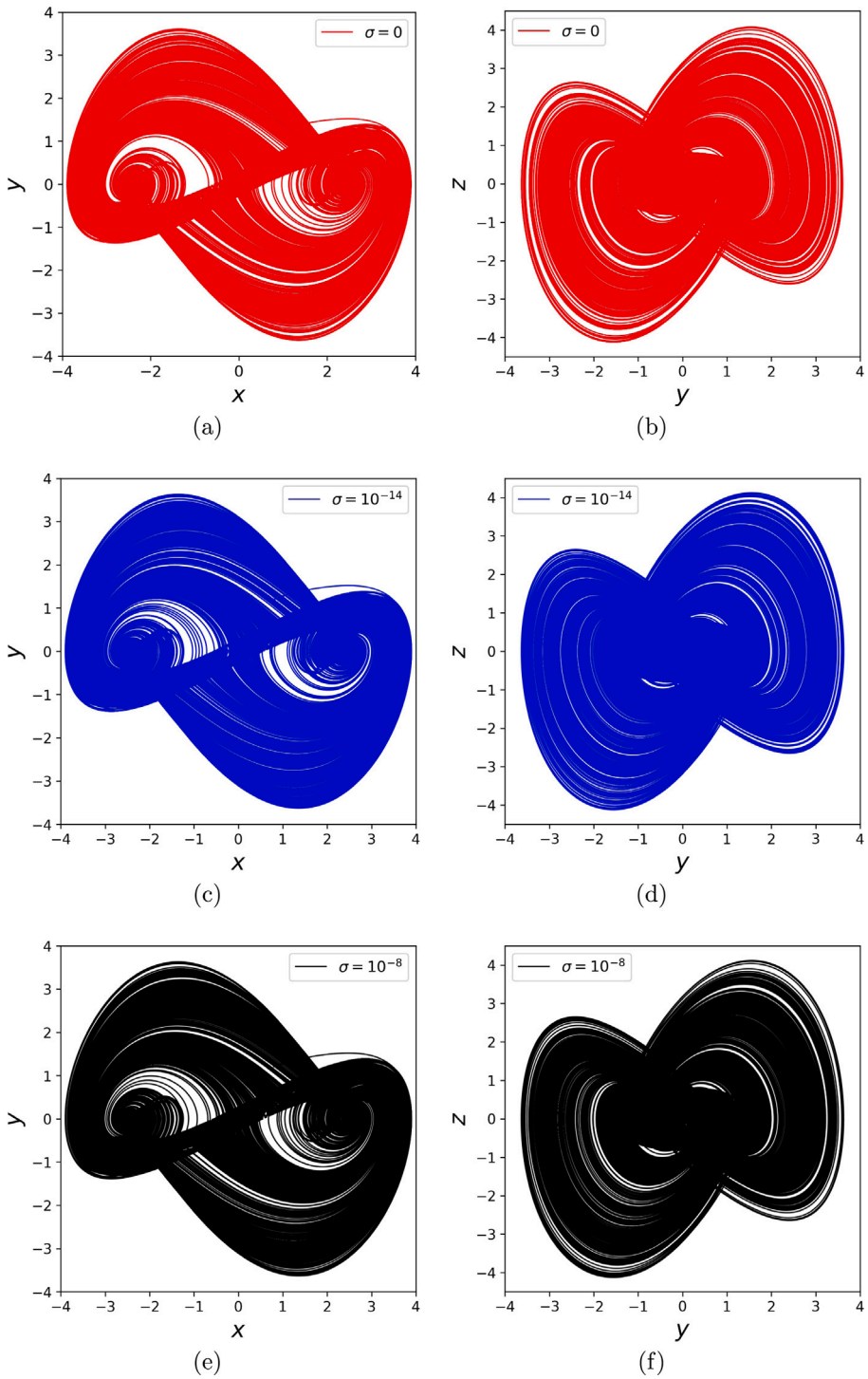

Fig. 2. Influence of the tiny disturbances on the phase plots of the Vaidyanathan jerk chaotic navigation system in mobile robot. (a), (c), and (e) are x–y planes; (b), (d), and (f) are the y–z planes. The curves are based on the CNS results in $t \in [0, 10000]$ of the normal-chaos governed by the 3D novel Vaidyanathan jerk chaotic system (4)–(6) with (10) and (11), in the cases of $\sigma = 0$ (red), $\sigma = 10^{-14}$ (blue), and $\sigma = 10^{-8}$ (black).

3.1.1. The Vaidyanathan chaotic system

Here, we investigate the influence of small disturbances on the 3D novel Vaidyanathan jerk chaotic system [38,41]. The 3D novel Vaidyanathan jerk chaotic system Eqs. (4)–(6) is applied as the navigation system of mobile robot in the cases without disturbance $\sigma = 0$, and with two different tiny disturbances $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$ in the initial condition (11), respectively. For each case, by means of the CNS, we gain a reliable chaotic simulation in $t \in [0, 10000]$. To further investigate the influence of disturbances on the statistical behaviors of the navigation system, the simulation of the navigation system without disturbances ($\sigma = 0$) obtained by the CNS is applied as the benchmark solution. The x–y planes and y–z planes of the attractors for the simulations of navigation system with disturbances and without disturbances are shown in Fig. 2 . It is found that the attractors of the simulations with tiny disturbances $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$ are similar to those of simulation without disturbances $\sigma = 0$. Besides, as shown in Fig. 3, the probability density functions (PDFs) of the variables in the cases with disturbances $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$ are close to those in

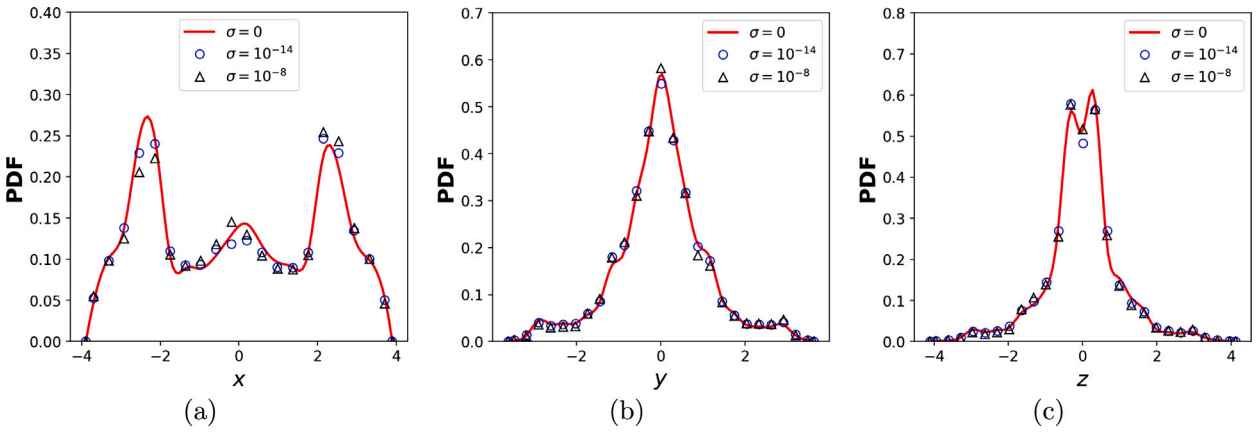

Fig. 3. Influence of tiny disturbances on the probability density functions (PDFs) of trajectories of the Vaidyanathan jerk navigation system in mobile robot. (a) The PDF of x; (b) The PDF of y; (c) The PDF of z. The curves are based on the CNS results in $t \in [0, 10000]$ of the normal-chaos governed by the 3D novel Vaidyanathan jerk chaotic system (4)–(6) with (10) and (11), in the cases of $\sigma = 0$ (red line), $\sigma = 10^{-14}$ (blue circle) and $\sigma = 10^{-8}$ (black triangle). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 1
Influence of tiny disturbances on the means (μ_x, μ_y, μ_z) and standard deviations $(\sigma_x, \sigma_y, \sigma_z)$ of the trajectories of the Vaidyanathan navigation system in chaotic mobile robot.

Vaidyanathan	μ_x	μ_y	μ_z	σ_x	σ_y	σ_z
$\sigma = 0$	−0.05	0.00	0.00	2.18	1.06	0.98
$\sigma = 10^{-14}$	0.00	0.00	0.00	2.19	1.08	1.00
$\sigma = 10^{-8}$	0.06	0.00	0.00	2.17	1.07	0.98

The means and standard deviations are based on the CNS results in $t \in [0, 10000]$ of the normal-chaos governed by the 3D novel Vaidyanathan jerk chaotic system (4)–(6) with (10) and (11), in the cases of $\sigma = 0$, $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$.

Table 2
Influence of tiny disturbances on the skewnesses (s_x, s_y, s_z) and kurtoses (k_x, k_y, k_z) of the trajectories of the Vaidyanathan navigation system in chaotic mobile robot.

Vaidyanathan	s_x	s_y	s_z	k_x	k_y	k_z
$\sigma = 0$	0.04	−0.07	−0.07	−1.39	1.15	1.97
$\sigma = 10^{-14}$	0.00	0.02	0.02	−1.42	1.11	1.92
$\sigma = 10^{-8}$	−0.05	0.08	0.07	−1.38	1.18	1.99

The skewnesses and kurtoses are based on the CNS results in $t \in [0, 10000]$ of the normal-chaos governed by the 3D novel Vaidyanathan jerk chaotic system (4)–(6) with (10) and (11), in the cases of $\sigma = 0$, $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$.

the case without disturbances $\sigma = 0$. The PDF [46] defines the density of probability for the variable that falls within a particular range of values. The integral of the variable's PDF over that range gives the probability. Thus, to compute the PDF from a given set of variable α with the range $\alpha \in [\alpha_{min}, \alpha_{max}]$, the whole interval can be divided into k small intervals with the length $\Delta\alpha = (\alpha_{max} - \alpha_{min})/k$. Then, the PDF $f(\alpha)$ is given as $f(\alpha) = N_\alpha/(N\Delta\alpha)$, where N_α is the number of samples falling in the interval $[\alpha - \Delta\alpha/2, \alpha + \Delta\alpha/2]$ and N is the number of all the samples. For the details of the computation of PDF, please refer to [46]. Additionally, as shown in Tables 1 and 2, the means, standard deviations, skewnesses, and kurtoses of the variables x, y, and z in the navigation system with $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$ are also close to the corresponding statistics with $\sigma = 0$. We have also studied the influence of the tiny disturbances on the 3D novel Vaidyanathan jerk chaotic system with a longer simulation time, i.e., $t \in [0, 30000]$. Based on the reliable simulations given by the CNS, the same conclusion has been obtained. That is, tiny disturbances do not affect the statistical properties of the 3D novel Vaidyanathan jerk chaotic system.

Therefore, although the tiny disturbances lead to huge deviations in the trajectories of the variables in the navigation system, these tiny disturbances do not have influence on the statistics. Thus, the 3D novel Vaidyanathan jerk chaotic system that is used as the navigation system of the mobile robot belongs to normal-chaos [24], whose statistics are insensitive to the tiny disturbances.

3.1.2. The hyper-chaotic Rössler chaotic system

Here, we consider the hyper-chaotic Rössler system Eqs. (14)–(16) as the navigation system of the mobile robot without disturbance $\sigma = 0$, and with two different tiny disturbances $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$ in the initial condition (22), respectively. It has been illustrated in Ref. [24] that the hyper-chaotic Rössler system belongs to normal-chaos as small disturbances do not affect its statistics. As illustrated in Section 2.2, for all three cases with different values of σ, we can gain the reliable chaotic simulations in $t \in [0, 10000]$ by means of the CNS. In Fig. 4, we present the results about the x–y plane of the simulations and the PDF of x without disturbance $\sigma = 0$, and with disturbances $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$. Table 3 shows the means and standard deviations of trajectories of variables in the navigation system for these three cases with different values of σ, and Table 4 presents the skewnesses and kurtoses. It is shown that the statistics of the hyper-chaotic Rössler system are insensitive to small disturbances. This confirms that the hyper-chaotic Rössler system belongs to normal-chaos [24], whose statistics are not affected by small disturbance.

3.2. Influence of tiny disturbances on robot motion

In this section, we further investigate the influence of tiny disturbances in the initial condition on the motion of the chaotic mobile robot with the 3D novel Vaidyanathan jerk chaotic navigation system and the hyper-chaotic Rössler navigation system, respectively.

3.2.1. Motion controlled by the Vaidyanathan chaotic system

First, we consider the influence of small disturbances in the initial condition (11) on the mobile robot motion with the Vaidyanathan jerk system controller. By means of the CNS, we gain a reliable chaotic simulation in $t \in [0, 10000]$, for each case of the chaotic mobile robot with $\sigma = 0$, $\sigma = 10^{-14}$, and $\sigma = 10^{-8}$, respectively. The simulation of the chaotic mobile robot without disturbances (i.e., $\sigma = 0$) obtained by the CNS is applied as the benchmark solution to further study the influence of disturbances on the motion of chaotic mobile robot. In Fig. 5(a), the chaotic motions of the mobile robot in $t \in [0, 10000]$ with different values of σ are shown. Due to the tiny disturbances, the trajectories of mobile robot with $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$ deviate from the true simulation with $\sigma = 0$. As shown in Fig. 6, the PDFs of the trajectories X and Y of the chaotic mobile robot with disturbances $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$ significantly deviate from those in the case of $\sigma = 0$. Table 5 present the means (μ_X, μ_Y), standard deviations (σ_X, σ_Y), skewnesses

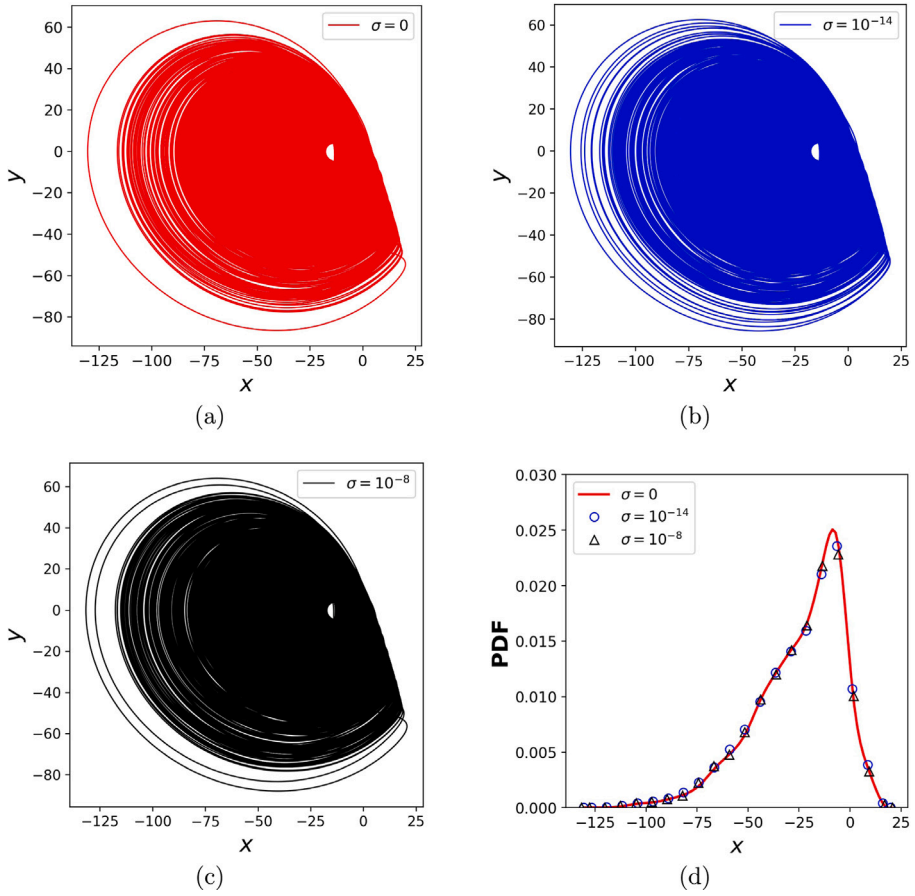

Fig. 4. Influence of the tiny disturbances on the phase plot and the PDF of the hyper-chaotic Rössler system. (a) x–y plane with $\sigma = 0$; (b) x–y plane with $\sigma = 10^{-14}$; (c) x–y plane with $\sigma = 10^{-8}$. (d) PDF of x. The curves are based on the CNS results in $t \in [0, 10000]$ of the normal-chaos governed by the hyper-chaotic Rössler system (14)–(17) with (21) and (22), in the cases of $\sigma = 0$ (red), $\sigma = 10^{-14}$ (blue) and $\sigma = 10^{-8}$ (black). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 3
Influence of tiny disturbances on the means $(\mu_x, \mu_y, \mu_z, \mu_\omega)$ and standard deviations $(\sigma_x, \sigma_y, \sigma_z, \sigma_\omega)$ of the trajectories of the hyper-chaotic Rössler navigation system in chaotic mobile robot.

Rössler	μ_x	μ_y	μ_z	μ_ω	σ_x	σ_y	σ_z	σ_ω
$\sigma = 0$	−24.87	−2.54	2.55	25.50	21.52	20.10	12.56	8.31
$\sigma = 10^{-14}$	−25.69	−2.63	2.63	26.35	22.26	20.79	13.02	8.59
$\sigma = 10^{-8}$	−25.44	−2.61	2.61	26.09	22.09	20.61	12.89	8.59

The means and standard deviations are based on the CNS results in $t \in [0, 10000]$ of the normal-chaos governed by the hyper-chaotic Rössler system (14)–(17) with (21) and (22), in the cases of $\sigma = 0$, $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$.

Table 4
Influence of tiny disturbances on the skewnesses $(s_x, s_y, s_z, s_\omega)$ and kurtoses $(k_x, k_y, k_z, k_\omega)$ of the trajectories of the hyper-chaotic Rössler navigation system in chaotic mobile robot.

Rössler	s_x	s_y	s_z	s_ω	k_x	k_y	k_z	k_ω
$\sigma = 0$	−0.94	−0.33	9.37	0.79	0.74	0.09	115.96	0.39
$\sigma = 10^{-14}$	−0.92	−0.33	9.54	0.76	0.70	0.07	120.71	0.28
$\sigma = 10^{-8}$	−0.94	−0.33	9.62	0.79	0.77	0.10	123.12	0.36

The skewnesses and kurtoses are based on the CNS results in $t \in [0, 10000]$ of the normal-chaos governed by the hyper-chaotic Rössler system (14)–(17) with (21) and (22), in the cases of $\sigma = 0$, $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$.

(s_X, s_Y) and kurtoses (k_X, k_Y) of X and Y of chaotic mobile robot in the cases of $\sigma = 0$, $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$. The statistics of the motion of chaotic mobile robot given by the CNS in the cases with tiny disturbances $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$ are obviously different from those in the case without disturbance $\sigma = 0$. If the simulation time is increased to 30000, the same qualitative conclusion can be obtained based on the reliable simulations given by the CNS. That is, the motion of mobile robot is sensitive to tiny disturbances, even in the statistical meaning.

Then, we compare the chaotic motions of mobile robot with $\sigma = 0, \sigma = 10^{-14}$, and $\sigma = 10^{-8}$ in the same dimension of workspace, i.e., $(X, Y) \in [0, 10] \times [0, 10]$. As shown in Fig. 5(b), the tiny disturbances in initial condition (11) lead to large deviation of mobile robot motion after some time. Fig. 7 presents the PDFs of trajectories X and Y for these three cases. Table 6 shows the means (μ_X, μ_Y), standard deviations (σ_X, σ_Y), skewnesses (s_X, s_Y), and kurtoses (k_X, k_Y) of X and Y for mobile robot motion. Thus, if the motion of mobile robot are compared in the same workspace, the statistics of robot motion with

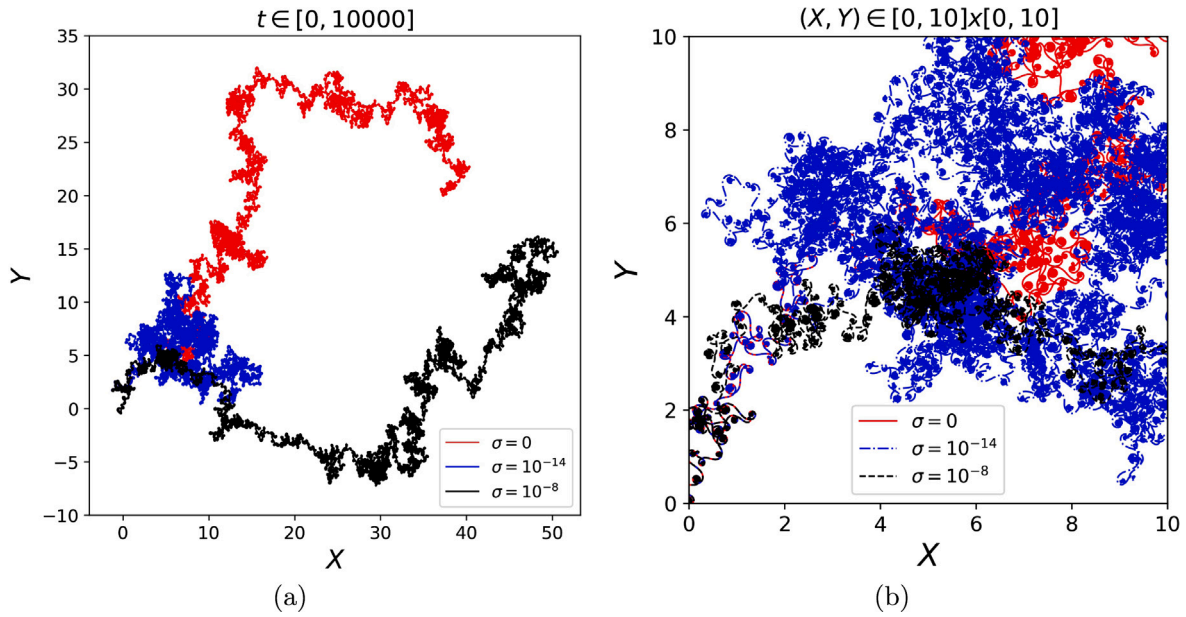

Fig. 5. Influence of tiny disturbances on the motion of chaotic mobile robot controlled by the Vaidyanathan system. The curves are based on the CNS results in $t \in [0, 10000]$ (a) and $(X, Y) \in [0, 10] \times [0, 10]$ (b) of the ultra-chaotic motions of mobile robot governed by the (4)–(9) with (10) and (11), in the cases of $\sigma = 0$ (red), $\sigma = 10^{-14}$ (blue), and $\sigma = 10^{-8}$ (black). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

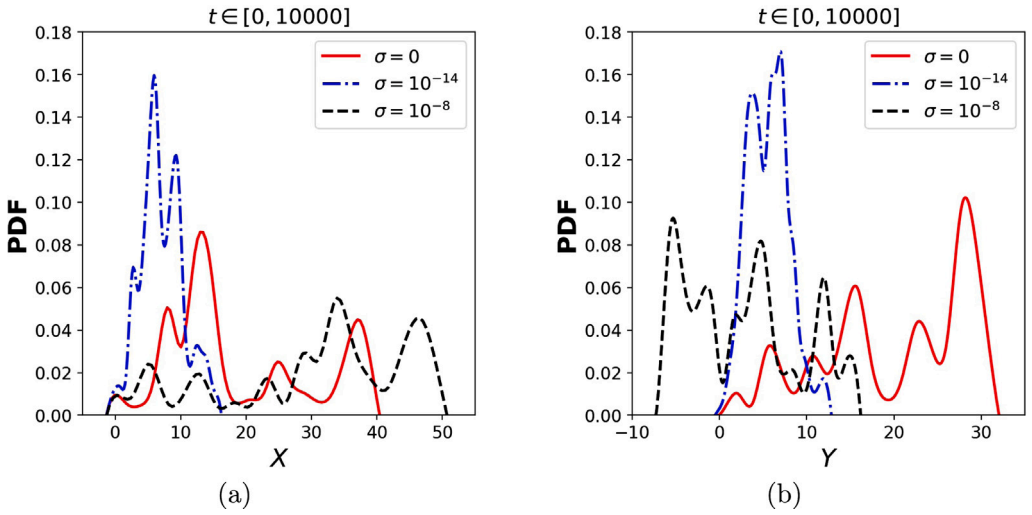

Fig. 6. Influence of tiny disturbances on the PDFs of trajectories X (a) and Y (b) of the chaotic mobile robot controlled by the Vaidyanathan system. The PDFs are based on the CNS results in $t \in [0, 10000]$ of the ultra-chaotic motions of mobile robot governed by the (4)–(9) with (10) and (11), in the cases of $\sigma = 0$ (red), $\sigma = 10^{-14}$ (blue dash-dotted), and $\sigma = 10^{-8}$ (black dashed). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

disturbances $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$ are also obviously different from the corresponding results without disturbance $\sigma = 0$. Therefore, tiny disturbances lead to huge differences not only in the trajectories but also the statistical properties. It is found that the motion of chaotic mobile robot belongs to ultra-chaos [24], whose statistics are even sensitive to tiny disturbances.

Furthermore, as shown in Fig. 8, we vary the parameters a and b, i.e. $a \in [0.30, 0.50]$, $b \in [0.60, 0.99]$, to find out whether this kind of ultra-chaos widely exists in the chaotic mobile robot with the Vaidyanathan jerk chaotic navigation system. For the parameters where the motion of mobile robot belongs to non-chaos, the navigation system with these parameters all belongs to non-chaos. Besides, for the parameters where the motion of chaotic mobile robot belongs to ultra-chaos, the navigation system with these parameters all belongs to normal-chaos. Although the trajectory of chaotic navigation system has sensitive dependence on tiny disturbances, its statistics are insensitive to the disturbances. However, the motion of the chaotic mobile robot is sensitive to small disturbances, even in the statistical meaning. The ultra-chaos governed by the chaotic mobile robot indeed widely exist.

3.2.2. Motion controlled by the hyper-chaotic Rössler system

Here, we consider the motion of mobile robot controlled by the hyper-chaotic Rössler system. The simulation of the chaotic mobile robot without disturbances (i.e., $\sigma = 0$) obtained by the CNS is applied as the benchmark solution to further study the influence of disturbances on the motion of chaotic mobile robot. In Fig. 9(a), the motions of the mobile robot in $t \in [0, 10000]$ with $\sigma = 0$, $\sigma = 10^{-14}$, and $\sigma = 10^{-8}$ are shown. Fig. 9(b) compares the motions of mobile robot in the same dimension of workspace $(X, Y) \in [-20, 20] \times [-20, 20]$. Due to the tiny disturbances in the initial condition, the trajectory of mobile robot deviates from the true simulation after a certain time. In

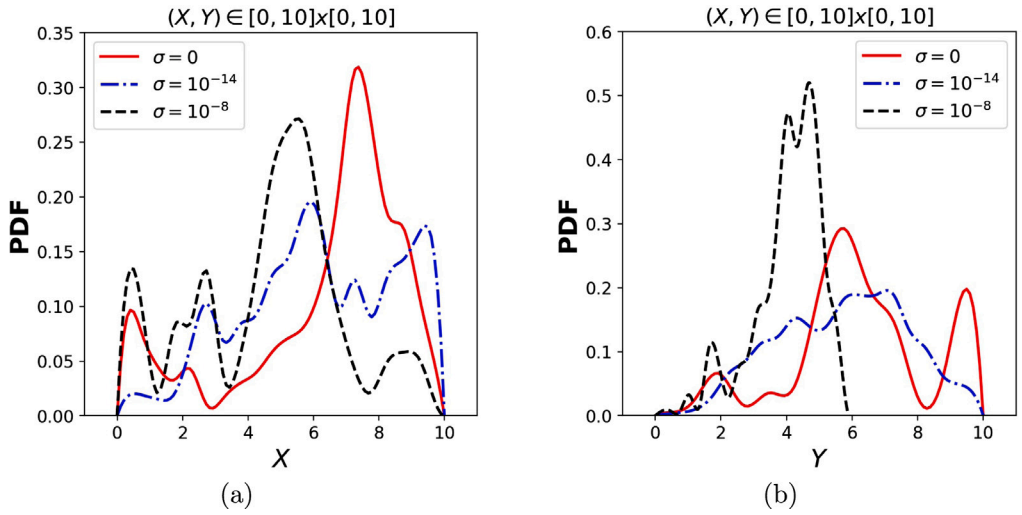

Fig. 7. Influence of tiny disturbances on the PDFs of trajectories X (a) and Y (b) of the chaotic mobile robot controlled by the Vaidyanathan system. The PDFs are based on the CNS results in $(X, Y) \in [0, 10] \times [0, 10]$ of the ultra-chaotic motions of mobile robot governed by the (4)–(9) with (10) and (11), in the cases of $\sigma = 0$ (red), $\sigma = 10^{-14}$ (blue dash-dotted), and $\sigma = 10^{-8}$ (black dashed). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

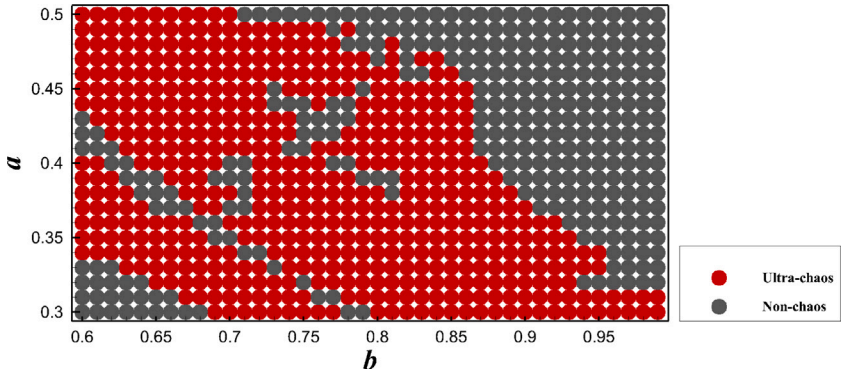

Fig. 8. Motion types of the mobile robot controlled by the Vaidyanathan system with different parameters a and b. Red dot: ultra-chaos; gray dot: non-chaos. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 5
Influence of tiny disturbances on the means (μ_X, μ_Y) and standard deviations (σ_X, σ_Y) of trajectories X and Y of the chaotic mobile robot.

Disturbance	μ_X	μ_Y	σ_X	σ_Y	s_X	s_Y	k_X	k_Y
$\sigma = 0$	18.93	20.21	11.10	8.37	0.56	−0.49	−1.00	−1.00
$\sigma = 10^{-14}$	7.16	5.62	3.30	2.34	0.32	0.34	−0.09	−0.12
$\sigma = 10^{-8}$	30.61	2.74	14.07	6.48	−0.66	0.33	−0.64	−1.06

The means and standard deviations are based on the CNS results in $t \in [0, 10000]$ of the ultra-chaotic motions of mobile robot governed by the (4)–(9) with (10) and (11), in the cases of $\sigma = 0$, $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$

Table 6
Influence of tiny disturbances on the means (μ_X, μ_Y), standard deviations (σ_X, σ_Y), skewnesses (s_X, s_Y), and kurtoses (k_X, k_Y) of trajectories X and Y of the chaotic mobile robot controlled by the Vaidyanathan system.

Vaidyanathan	μ_X	μ_Y	σ_X	σ_Y	s_X	s_Y	k_X	k_Y
$\sigma = 0$	6.34	6.28	2.54	2.15	−1.17	−0.21	0.37	−0.03
$\sigma = 10^{-14}$	6.18	5.71	2.35	1.96	−0.28	−0.14	−0.74	−0.69
$\sigma = 10^{-8}$	4.64	4.06	2.29	1.02	−0.19	−1.09	−0.43	1.07

The statistics are based on the CNS results in $(X, Y) \in [0, 10] \times [0, 10]$ of the ultra-chaotic motions of mobile robot with the Vaidyanathan navigation system governed by the (4)–(9) with (10) and (11), in the cases of $\sigma = 0$, $\sigma = 10^{-14}$, and $\sigma = 10^{-8}$.

Fig. 10, the PDFs of the trajectories X and Y of the chaotic mobile robot with disturbances $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$ are significantly different from those in the case of $\sigma = 0$, based on the simulations in the same time interval $t \in [0, 10000]$. As shown in Fig. 11, we have the similar conclusion when the PDFs are compared based on the motions of chaotic mobile robot in the same workspace $(X, Y) \in [-20, 20] \times [-20, 20]$.

Table 7 presents the means (μ_X, μ_Y), standard deviations (σ_X, σ_Y), skewnesses (s_X, s_Y), and kurtoses (k_X, k_Y) of X and Y of chaotic mobile robot in the cases of $\sigma = 0$, $\sigma = 10^{-14}$, and $\sigma = 10^{-8}$ for $t \in [0, 10000]$.

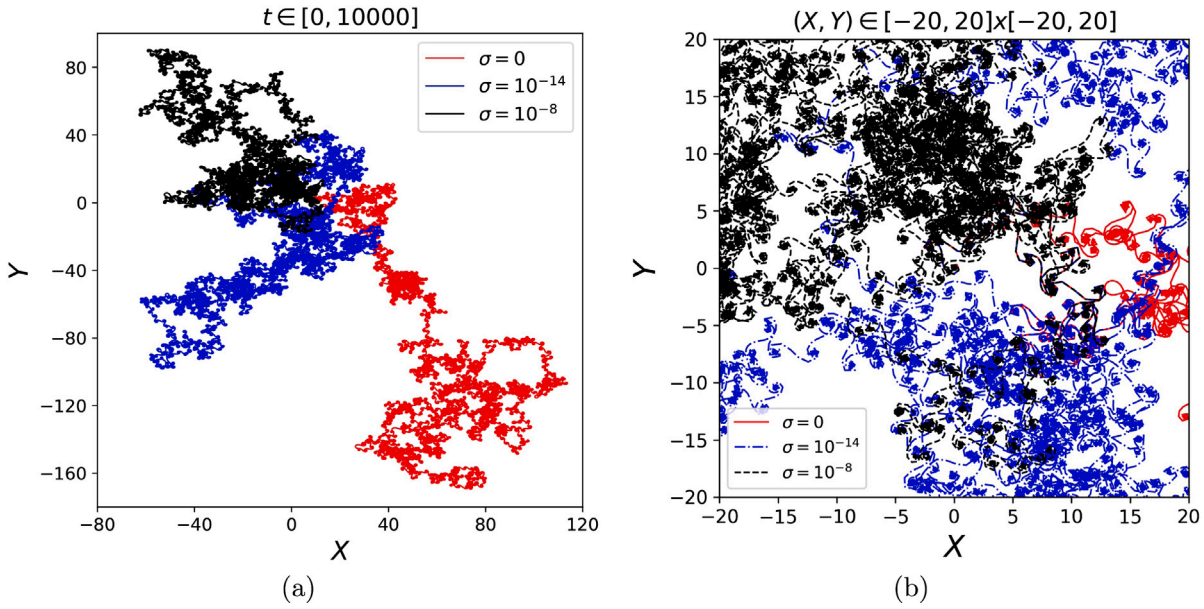

Fig. 9. Influence of tiny disturbances on the motion of chaotic mobile robot controlled by the Rössler system. The curves are based on the CNS results in $t \in [0, 10000]$ (a) and $(X, Y) \in [-20, 20] \times [-20, 20]$ (b) of the ultra-chaotic motions of mobile robot governed by the (14)–(20) with (21) and (22), in the cases of $\sigma = 0$ (red), $\sigma = 10^{-14}$ (blue), and $\sigma = 10^{-8}$ (black). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

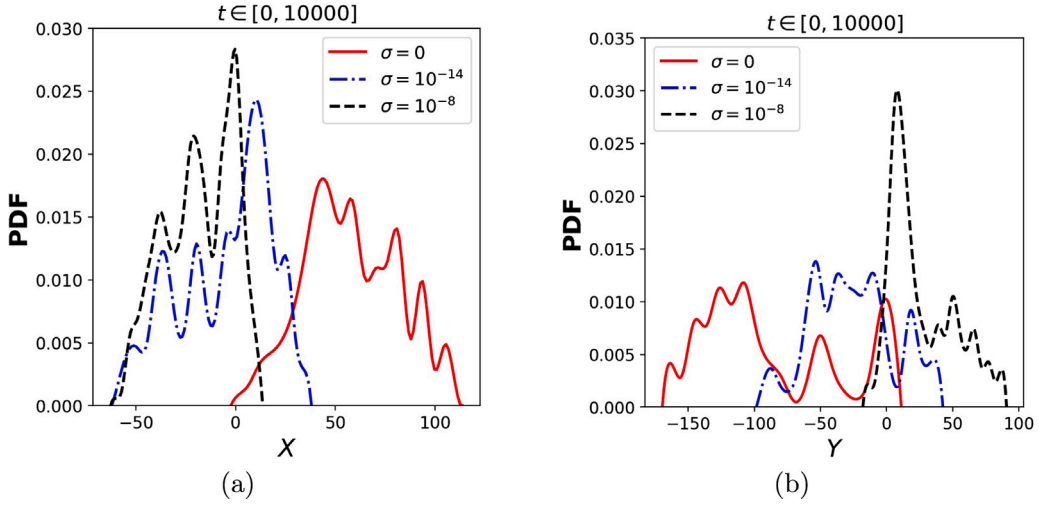

Fig. 10. Influence of tiny disturbances on the PDFs of trajectories X (a) and Y (b) of the chaotic mobile robot controlled by the Rössler system. The PDFs are based on the CNS results in $t \in [0, 10000]$ of the ultra-chaotic motions of mobile robot governed by the (14)–(20) with (21) and (22), in the cases of $\sigma = 0$ (red), $\sigma = 10^{-14}$ (blue dash-dotted), and $\sigma = 10^{-8}$ (black dashed). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Besides, Table 8 describes these statistics computed from simulations of motion in the same workspace i.e., $(X, Y) \in [-20, 20] \times [-20, 20]$.

The statistics of the motion of chaotic mobile robot given by the CNS in the cases with tiny disturbances $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$ are obviously different from those in the case without disturbances $\sigma = 0$. Thus, tiny disturbances lead to the huge differences, even in the statistical properties of the motion of chaotic mobile robot. Therefore, for the motion of mobile robot controlled by the hyper-chaotic Rössler system, we have the same qualitative result that the motion of chaotic mobile robot is sensitive to the tiny disturbances, even in the statistical meaning. The motion of the mobile robot navigated by the hyper-chaotic Rössler system also belongs to ultra-chaos.

Note that, the trajectory of mobile robot with the chaotic navigation system is not a random walk. For the time series of a random walk, its first-order difference belongs to white noise. Through the Ljung–Box test [47], we examine whether the first-order differences of trajectories of mobile robot, i.e. ΔX and ΔY, belong to white noises. As the p-values from the Ljung–Box test for the cases with small disturbances $\sigma = 0$, $\sigma = 10^{-14}$ and $\sigma = 10^{-8}$ in the initial condition are all 0.00 less than 0.05. Thus, the first-order differences of trajectories have autocorrelations, and they are not white noises. The trajectory of chaotic mobile robot is sensitive to the initial condition, while it is not a random walk.

In practice, since there always exist random environmental noises and they are out of control, the trajectories would deviate from the true numerical results after a short period of time. It is difficult to replicate the true long-term numerical results of the ultra-chaos governed by the chaotic mobile robot, even in the statistical meanings. The ultra-chaos governed by the chaotic mobile robot is an objective obstacle to reproducibility. Therefore, the ultra-chaos is indeed a higher-level disorder compared with the normal-chaos.

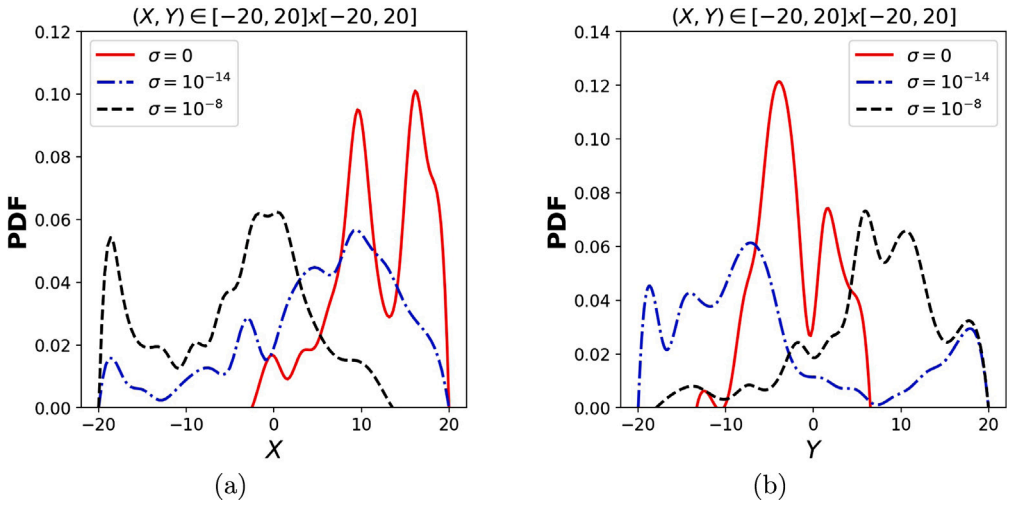

Fig. 11. Influence of tiny disturbances on the PDFs of trajectories X (a) and Y (b) of the chaotic mobile robot controlled by the Rössler system. The PDFs are based on the CNS results in $(X, Y) \in [-20, 20] \times [-20, 20]$ of the ultra-chaotic motions of mobile robot governed by the (14)–(20) with (21) and (22), in the cases of $\sigma = 0$ (red), $\sigma = 10^{-14}$ (blue dash-dotted), and $\sigma = 10^{-8}$ (black dashed). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 7
Influence of tiny disturbances on the means (μ_X, μ_Y), standard deviations (σ_X, σ_Y), skewnesses (s_X, s_Y), and kurtoses (k_X, k_Y) of trajectories X and Y of the chaotic mobile robot controlled by the Rössler system.

Rössler	μ_X	μ_Y	σ_X	σ_Y	s_X	s_Y	k_X	k_Y
$\sigma = 0$	57.76	−88.24	23.71	52.07	0.08	0.54	−0.62	−1.02
$\sigma = 10^{-14}$	−5.07	−25.52	23.31	31.43	−0.47	0.12	−0.81	−0.56
$\sigma = 10^{-8}$	−18.33	26.33	17.21	25.02	−0.30	0.75	−0.95	−0.52

The statistics are based on the CNS results in $t \in [0, 10000]$ of the ultra-chaotic motions of mobile robot governed by the (14)–(20) with (21) and (22), in the cases of $\sigma = 0, \sigma = 10^{-14}$, and $\sigma = 10^{-8}$.

Table 8
Influence of tiny disturbances on the means (μ_X, μ_Y), standard deviations (σ_X, σ_Y), skewnesses (s_X, s_Y), and kurtoses (k_X, k_Y) of trajectories X and Y of the chaotic mobile robot controlled by the Rössler system.

Rössler	μ_X	μ_Y	σ_X	σ_Y	s_X	s_Y	k_X	k_Y
$\sigma = 0$	12.00	−2.15	5.24	3.95	−0.57	0.21	−0.40	−0.71
$\sigma = 10^{-14}$	5.11	−4.97	9.07	11.19	−0.88	0.90	0.36	−0.27
$\sigma = 10^{-8}$	−4.43	6.94	8.47	7.56	−0.30	−0.85	−0.85	0.76

The statistics are based on the CNS results in $(X, Y) \in [-20, 20] \times [-20, 20]$ of the ultra-chaotic motions of mobile robot governed by the (14)–(20) with (21) and (22), in the cases of $\sigma = 0$, $\sigma = 10^{-14}$, and $\sigma = 10^{-8}$.

4. Concluding remarks

According to the famous butterfly-effect, trajectories of all chaotic dynamical systems are rather sensitive to tiny disturbances. Note that a new concept, called ultra-chaos [24], has been recently proposed. The ultra-chaos is quite different from the normal-chaos whose statistics are the same even when there exist disturbances. The statistics of an ultra-chaos are even sensitive to small disturbances. It is a pity that few ultra-chaos have been reported. In the present paper, using the reliable simulation given by the CNS without disturbances as the benchmark solution, we study the influences of small disturbances on the navigation system and the motion of a chaotic mobile robot, respectively.

It is illustrated that the chaotic navigation system of the mobile robot belongs to normal-chaos. Although tiny disturbances lead to huge deviations in the trajectories of the chaotic navigation system, they do not affect the statistics. However, it is found that the motion of the chaotic mobile robot belongs to ultra-chaos. The statistical properties of the chaotic mobile robot have sensitivity dependence on tiny disturbances. The small disturbances could lead to the loss of the reproducibility of the motion of chaotic mobile robot, even in statistical meanings. Thus, the ultra-chaos obviously represents a higher-level of disorder compared with the normal-chaos, and it widely exists in practice. Hopefully, the ultra-chaos as a new concept could deepen our understandings of chaos and provide us with a new door to explore chaos theory.

CRediT authorship contribution statement

Yu Yang: Methodology, Software, Writing – original draft. **Shijie Qin:** Investigation, Formal analysis, Data curation. **Shijun Liao:** Conceptualization, Methodology, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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