Series solutions of coupled Van der Pol equation by means of homotopy analysis method

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In this paper, the homotopy analysis method (HAM) is used to give series solutions of self-exited oscillation systems governed by two Van der Pol equations, which are coupled by a linear and a cubic term. The frequency and amplitude of all possible periodic solutions are investigated. It is found that there exist either in-phase or out-of-phase periodic solutions only. Besides, the in-phase periodic oscillations are decoupled, whose periods and amplitudes have nothing to do with the linear and cubic coupled terms. However, the out-of-phase periodic oscillations are strongly coupled, whose period and amplitude can be controlled by the linear and cubic coupled terms. © *2010 American Institute of Physics.* [doi:10.1063/1.3445770]

I. INTRODUCTION

Van der Pol equation[1] provides an example of a oscillator with nonlinear damping, energy being dissipated at large amplitudes and generated at low amplitudes. Such systems typically possess limit cycles: sustained oscillations around a state at which energy generation and dissipation balance. The original application described by Van der Pol[1] models an electrical circuit with a triode valve, the resistive properties of which change with current, the low current, negative resistance becoming positive as current increases. This model has been widely applied in science and engineering.[2–6]

Currently, Nohara and Arimoto[7] considered the oscillation system of two linearly coupled Van der Pol equations. They proved that there exist either in-phase or out-of-phase periodic solutions only. In this paper, we further investigate the existence of periodic solutions of two nonlinearly coupled Van der Pol equations by means of the homotopy analysis method (HAM).[8–14] Unlike perturbation techniques,[15–18] the HAM is independent of any small physical parameters. More importantly, the HAM provides a simple way to ensure the convergence of solution series so that one can always get accurate enough approximations even for strongly nonlinear problems. Furthermore, the HAM provides great freedom to choose the so-called auxiliary linear operator so that one can approximate a nonlinear problem more effectively by means of better base functions, as illustrated by Liao and Tan.[13] This kind of freedom is so large that the second-order nonlinear two-dimensional Gelfand equation can be solved even by means of a fourth-order auxiliary linear operator in the frame of the HAM, as shown in Ref. 13. Especially, by means of the HAM, a few new solutions of some nonlinear problems are found,[19,20] which are neglected by all other analytic methods and even by numerical techniques. Currently, the relationship between the HAM and the famous Euler transform was revealed.[21] The HAM has been widely applied to solve many nonlinear problems in science, engineering, and finance.[22–32] All of these show the potential of the HAM.

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II. THE HAM APPROACH

Let us consider the periodic solutions of the two coupled Van der Pol equations,

$$\begin{cases} \ddot{y} + \varepsilon(y^2 - 1)\dot{y} + y = k(y - z) + \mu(y^3 - z^3) \\ \ddot{z} + \varepsilon(z^2 - 1)\dot{z} + z = k(z - y) + \mu(z^3 - y^3), \end{cases} \tag{1}$$

where the dot denotes the derivative with respect to the time t, ε is a physical parameter for typical Van der Pol oscillations, and k and μ are the linear and nonlinear couple parameters, respectively. It is well known that free oscillations of self-excited systems have periodic limit cycles independent of initial conditions. Periodic solutions of self-exited systems contain two important physical parameters, i.e., the angular frequency ω and the amplitude α.

First of all, let us assume that the two periodic solutions $y(t)$ and $z(t)$ have the same period. Then, without loss of generality, we consider the initial conditions,

$$y(0) = \alpha, \quad \dot{y}(0) = 0, \quad z(0) = c\alpha, \quad \dot{z}(0) = \alpha\omega d, \tag{2}$$

where ω and α denote the angular frequency and amplitude of the periodic solution $y(t)$, and the amplitude of the periodic solution $z(t)$ is determined by c and d, respectively. Note that ω, α, c, and d are unknown constants.

Let $\tau=\omega t$ denote a new time scale. Under the transformation

$$\tau = \omega t, \quad y(t) = \alpha u(\tau), \quad z(t) = \alpha v(\tau), \tag{3}$$

the two original equations (1) and its initial conditions (2) become

$$\begin{cases} \omega^2 u'' + \omega\varepsilon(\alpha^2 u^2 - 1)u' + u = k(u - v) + \mu\alpha^2(u^3 - v^3) \\ \omega^2 v'' + \omega\varepsilon(\alpha^2 v^2 - 1)v' + v = k(v - u) + \mu\alpha^2(v^3 - u^3) \end{cases} \tag{4}$$

and

$$u(0) = 1, \quad u'(0) = 0, \quad v(0) = c, \quad v'(0) = d, \tag{5}$$

where the prime denotes the derivative with respect to τ. The above coupled nonlinear differential equations can be solved by means of the HAM,[11] as shown in the following sections.

Obviously, the periodic solutions of the two coupled self-exited oscillation systems $u(\tau)$ and $v(\tau)$ can be expressed by a set of periodic functions,

$$\{\cos(n\tau), \sin(n\tau) | n = 1, 2, \ldots\}$$

that

$$u(\tau) = \sum_{n=1}^{+\infty} [a_n \cos(n\tau) + b_n \sin(n\tau)], \quad v(\tau) = \sum_{n=1}^{+\infty} [\beta_n \cos(n\tau) + \gamma_n \sin(n\tau)], \tag{6}$$

where $a_n, b_n, \beta_n, \gamma_n$ are constant coefficients. This provides us the so-called solution expressions of $u(\tau)$ and $v(\tau)$, respectively.

Let ω_0, α_0 denote the initial approximation of the frequency ω and the amplitude α, respectively. Considering initial conditions (5) and solution expression (6), it is natural to choose the initial guess,

$$u_0(\tau) = \cos(\tau), \quad v_0(\tau) = c_0 \cos(\tau) + d_0 \sin(\tau), \tag{7}$$

where c_0 and d_0 are initial approximations of the unknown parameters c and d. Note that c_0 and d_0 are unknown now.

To obey solution expression (6), we choose such an auxiliary linear operator,

$$\mathcal{L}[\Phi(\tau;q)] = \omega_0^2\left[\frac{\partial^2\Phi(\tau;q)}{\partial\tau^2} + \Phi(\tau;q)\right], \tag{8}$$

where ω_0 is an initial guess of the frequency but is unknown right now. Obviously, the above operator has the property

$$\mathcal{L}(C_1 \sin \tau + C_2 \cos \tau) = 0,$$

where C_1, C_2 are constant coefficients.

According to Eq. (4), we define such two nonlinear operators,

$$\mathcal{N}_u[U(\tau;q),V(\tau;q),\Omega(q),A(q)] = \Omega^2(q)\frac{\partial^2 U(\tau;q)}{\partial\tau^2} + \varepsilon\Omega(q)(A^2(q)U^2(\tau;q) - 1)\frac{\partial U(\tau;q)}{\partial\tau} + U(\tau;q) - k(U(\tau;q) - V(\tau;q)) - \mu A^2(q)(U^3(\tau;q) - V^3(\tau;q)), \tag{9}$$

$$\mathcal{N}_v[V(\tau;q),U(\tau;q),\Omega(q),A(q)] = \Omega^2(q)\frac{\partial^2 V(\tau;q)}{\partial\tau^2} + \varepsilon\Omega(q)(A^2(q)V^2(\tau;q) - 1)\frac{\partial V(\tau;q)}{\partial\tau} + V(\tau;q) - k(V(\tau;q) - U(\tau;q)) - \mu A^2(q)(V^3(\tau;q) - U^3(\tau;q)). \tag{10}$$

Then, let $q \in [0,1]$ denote the embedding parameter and $\hbar$ a nonzero auxiliary parameter (called convergence-control parameter). We construct the so-called zeroth-order deformation equation,

$$(1-q)\mathcal{L}[U(\tau;q) - u_0(\tau)] = q\hbar\mathcal{N}_u[U(\tau;q),V(\tau;q),\Omega(q),A(q)], \tag{11}$$

$$(1-q)\mathcal{L}[V(\tau;q) - v_0(\tau)] = q\hbar\mathcal{N}_v[V(\tau;q),U(\tau;q),\Omega(q),A(q)], \tag{12}$$

subject to the initial conditions,

$$U(0;q) = 1, \quad \left.\frac{\partial U(\tau;q)}{\partial\tau}\right|_{\tau=0} = 0, \tag{13}$$

$$V(0;q) = C(q), \quad \left.\frac{\partial V(\tau;q)}{\partial\tau}\right|_{\tau=0} = D(q). \tag{14}$$

Obviously, when $q=0$, Eqs. (11)–(14) have the solution

$$U(\tau;0) = u_0(\tau) = \cos(\tau), \quad V(\tau;0) = v_0(\tau) = c_0\cos(\tau) + d_0\sin(\tau).$$

When $q=1$, Eqs. (11)–(14) are equivalent to original equations (4) and (5), provided

$$U(\tau;1) = u(\tau), \quad V(\tau;1) = v(\tau), \tag{15}$$

$$\Omega(1) = \omega, \quad A(1) = \alpha, \quad C(1) = c, \quad D(1) = d. \tag{16}$$

Therefore, as the embedding parameter q increases from 0 to 1, $U(\tau;q)$ and $V(\tau;q)$ vary from the initial guesses $u_0(\tau)$ and $v_0(\tau)$ to the exact solution $u(\tau)$ and $v(\tau)$, so does $\Omega(q)$ from ω_0 to ω, $A(q)$ from α_0 to α, $C(q)$ from c_0 to c, and $D(q)$ from d_0 to d, respectively.

Expand $U(\tau;q), V(\tau;q), \Omega(q), A(q), C(q), D(q)$ in the Maclaurin series with respect to q as follows:

$$U(\tau;q) = \sum_{n=0}^{+\infty} u_n(\tau)q^n, \quad V(\tau;q) = \sum_{n=0}^{+\infty} v_n(\tau)q^n, \tag{17}$$

$$\Omega(q) = \sum_{n=0}^{+\infty} \omega_n q^n, \quad A(q) = \sum_{n=0}^{+\infty} \alpha_n q^n, \tag{18}$$

$$C(q) = \sum_{n=0}^{+\infty} c_n q^n, \quad D(q) = \sum_{n=0}^{+\infty} d_n q^n, \tag{19}$$

where

$$u_n(\tau) = \frac{1}{n!} \left. \frac{\partial^n U(\tau;q)}{\partial q^n} \right|_{q=0}, \quad v_n(\tau) = \frac{1}{n!} \left. \frac{\partial^n V(\tau;q)}{\partial q^n} \right|_{q=0},$$

$$\omega_n = \frac{1}{n!} \left. \frac{\partial^n \Omega(q)}{\partial q^n} \right|_{q=0}, \quad \alpha_n = \frac{1}{n!} \left. \frac{\partial^n A(q)}{\partial q^n} \right|_{q=0},$$

$$c_n = \frac{1}{n!} \left. \frac{\partial^n C(q)}{\partial q^n} \right|_{q=0}, \quad d_n = \frac{1}{n!} \left. \frac{\partial^n D(q)}{\partial q^n} \right|_{q=0}.$$

Notice that all of the above series contain the convergence-control parameter $\hbar$, whose value we have freedom to choose. Assume that the convergence-control parameter $\hbar$ is so properly chosen that all these series are convergent at $q=1$. Thus, due to (15) and (16), one has at $q=1$ the homotopy-series solutions,

$$u(\tau) = \sum_{n=0}^{+\infty} u_n(\tau), \tag{20}$$

$$v(\tau) = \sum_{n=0}^{+\infty} v_n(\tau), \tag{21}$$

$$\omega = \sum_{n=0}^{+\infty} \omega_n, \tag{22}$$

$$\alpha = \sum_{n=0}^{+\infty} \alpha_n, \tag{23}$$

$$c = \sum_{n=0}^{+\infty} c_n, \tag{24}$$

$$d = \sum_{n=0}^{+\infty} d_n. \tag{25}$$

The mth-order approximation are given by

$$u(\tau) \approx \sum_{n=0}^{m} u_n(\tau), \quad v(\tau) \approx \sum_{n=0}^{m} v_n(\tau), \tag{26}$$

$$\omega \approx \sum_{n=0}^{m} \omega_n, \quad \alpha \approx \sum_{n=0}^{m} \alpha_n, \quad c \approx \sum_{n=0}^{m} c_n, \quad d \approx \sum_{n=0}^{m} d_n. \tag{27}$$

For the sake of simplicity, we can define such vectors,

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \ldots, u_n(\tau)\}, \quad \vec{v}_n = \{v_0(\tau), v_1(\tau), \ldots, v_n(\tau)\},$$

$$\vec{\alpha}_n = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}, \quad \vec{\omega}_n = \{\omega_0, \omega_1, \ldots, \omega_n\},$$

$$\vec{c}_n = \{c_0, c_1, \ldots, c_n\}, \quad \vec{d}_n = \{d_0, d_1, \ldots, d_n\}.$$

Differentiating n time equations (11)–(14) with respect to q and then setting $q=0$ and finally dividing them by $n!$, we have the nth-order deformation equation,

$$\mathcal{L}[u_n(\tau) - \chi_n u_{n-1}(\tau)] = \hbar \mathcal{R}_n^u(\vec{u}_{n-1}, \vec{v}_{n-1}, \vec{\omega}_{n-1}, \vec{\alpha}_{n-1}), \tag{28}$$

$$\mathcal{L}[v_n(\tau) - \chi_n v_{n-1}(\tau)] = \hbar \mathcal{R}_n^v(\vec{v}_{n-1}, \vec{u}_{n-1}, \vec{\omega}_{n-1}, \vec{\alpha}_{n-1}), \tag{29}$$

subject to the initial conditions

$$u_n(0) = 0, \quad u_n'(0) = 0, \quad v_n(0) = c_n, \quad v_n'(0) = d_n, \tag{30}$$

where

$$\chi_n = \begin{cases} 0, & n \leq 1 \\ 1, & n > 1, \end{cases}$$

and

$$\mathcal{R}_n^u(\vec{u}_{n-1}, \vec{v}_{n-1}, \vec{\omega}_{n-1}, \vec{\alpha}_{n-1}) = \sum_{i=0}^{n-1} u''_{n-1-i} W_i + u_{n-1} - \varepsilon F_{n-1}^u + \varepsilon \sum_{i=0}^{n-1} \Delta_i^u F_{n-1-i}^u - k(u_{n-1} - v_{n-1}) \tag{31}$$

$$- \mu\left(\sum_{i=0}^{n-1} \Delta_i^u u_{n-1-i} - \sum_{i=0}^{n-1} \Delta_i^v v_{n-1-i} \right), \tag{32}$$

$$\mathcal{R}_n^v(\vec{u}_{n-1}, \vec{v}_{n-1}, \vec{\omega}_{n-1}, \vec{\alpha}_{n-1}) = \sum_{i=0}^{n-1} v''_{n-1-i} W_i + v_{n-1} - \varepsilon F_{n-1}^v + \varepsilon \sum_{i=0}^{n-1} \Delta_i^v F_{n-1-i}^v - k(v_{n-1} - u_{n-1}) \tag{33}$$

$$- \mu\left(\sum_{i=0}^{n-1} \Delta_i^v v_{n-1-i} - \sum_{i=0}^{n-1} \Delta_i^u u_{n-1-i} \right), \tag{34}$$

with the definitions

$$W_i = \sum_{j=0}^{i} \omega_j \omega_{i-j}, \quad F_i^u = \sum_{j=0}^{i} \omega_j u'_{i-j}, \quad F_i^v = \sum_{j=0}^{i} \omega_j v'_{i-j},$$

$$A_r^u = \sum_{i=0}^{r} \alpha_i u_{r-i}, \quad A_r^v = \sum_{i=0}^{r} \alpha_i v_{r-i},$$

$$\Delta_i^u = \sum_{j=0}^{i} A_j^u A_{i-j}^u, \quad \Delta_i^v = \sum_{j=0}^{i} A_j^v A_{i-j}^v.$$

Notice that ω_{n-1}, α_{n-1}, c_{n-1}, and d_{n-1} are unknown now. Due to the rule of solution expression (6) and definition (8) of $\mathcal{L}$, solutions of nth-order deformation equations (28)–(30) should not contain the so-called secular terms $\tau\sin(\tau)$ and $\tau\cos(\tau)$. To ensure so, the right-hand side term $\mathcal{R}_n^u$ and $\mathcal{R}_n^v$ should not contain the terms $\sin(\tau)$ and $\cos(\tau)$. Therefore, the coefficients of $\sin(\tau)$ and $\cos(\tau)$ must be zero. So, write the right-hand term as follows:

$$\mathcal{R}_n^u(\tau) = \sum_{i=0}^{2n+1} [e_{n,i}\cos(i\tau) + f_{n,i}\sin(i\tau)], \tag{35}$$

$$\mathcal{R}_n^v(\tau) = \sum_{j=0}^{2n+1} [g_{n,j}\cos(j\tau) + h_{n,j}\sin(j\tau)], \tag{36}$$

where the coefficients $e_{n,i}$, $f_{n,i}$, $g_{n,j}$, and $h_{n,j}$ are given by

$$e_{n,i} = \frac{1}{\pi}\int_0^{2\pi} \mathcal{R}_n^u(\tau)\cos(i\tau)d\tau, \quad f_{n,i} = \frac{1}{\pi}\int_0^{2\pi} \mathcal{R}_n^u(\tau)\sin(i\tau)d\tau,$$

$$g_{n,j} = \frac{1}{\pi}\int_0^{2\pi} \mathcal{R}_n^v(\tau)\cos(j\tau)d\tau, \quad h_{n,j} = \frac{1}{\pi}\int_0^{2\pi} \mathcal{R}_n^v(\tau)\sin(j\tau)d\tau.$$

When $i=1$ and $j=1$, one gains four algebraic equations,

$$e_{n,1}(\vec{c}_{n-1},\vec{d}_{n-1},\vec{\omega}_{n-1},\vec{\alpha}_{n-1}) = 0, \quad f_{n,1}(\vec{c}_{n-1},\vec{d}_{n-1},\vec{\omega}_{n-1},\vec{\alpha}_{n-1}) = 0, \tag{37}$$

$$g_{n,1}(\vec{c}_{n-1},\vec{d}_{n-1},\vec{\omega}_{n-1},\vec{\alpha}_{n-1}) = 0, \quad h_{n,1}(\vec{c}_{n-1},\vec{d}_{n-1},\vec{\omega}_{n-1},\vec{\alpha}_{n-1}) = 0. \tag{38}$$

which determine the unknown α_{n-1}, ω_{n-1}, c_{n-1}, and $d_{n-1}(n\geq 1)$. Thereafter, it is easy to gain the solution of linear equations (28)–(30). In this way, one can gain $u_n(\tau)$, $v_n(\tau)$, ω_{n-1}, α_{n-1}, c_{n-1} and $d_{n-1}(n=1,2,3,\ldots)$, successfully.

When $n=1$, we have

$$R_1^u = e_{1,1}\cos(\tau) + e_{1,3}\cos(3\tau) + f_{1,1}\sin(\tau) + f_{1,3}\sin(3\tau),$$

$$R_1^v = g_{1,1}\cos(\tau) + g_{1,3}\cos(3\tau) + h_{1,1}\sin(\tau) + h_{1,3}\sin(3\tau).$$

To avoid the so-called secular terms, we must have

$$e_{1,1} = 0, \quad f_{1,1} = 0, \quad g_{1,1} = 0, \quad h_{1,1} = 0,$$

i.e.,

$$-\omega_0^2 + 1 - k + kc_0 + \tfrac{3}{4}\mu\alpha_0^2(c_0^3 + c_0 d_0^2 - 1) = 0,$$

$$-\tfrac{1}{4}\varepsilon\alpha_0^2\omega_0 + \varepsilon\omega_0 + kd_0 + \tfrac{3}{4}\mu\alpha_0^2(c_0^2 d_0 + d_0^3) = 0,$$

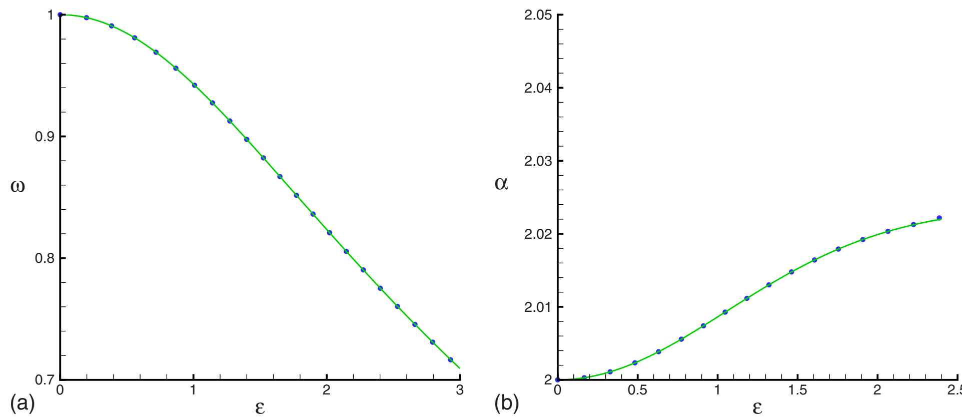

FIG. 1. (Color online) The curve $\varepsilon \sim \omega$ and $\varepsilon \sim \alpha$ by HAM-Padé approximation for the in-phase case. Filled circles: [10,10] HAM-Padé approximation; line: [15,15] HAM-Padé approximation.

$$-\omega_0^2 c_0 + \tfrac{1}{4}\varepsilon \alpha_0^2 \omega_0 c_0^2 d_0 + \tfrac{1}{4}\varepsilon \alpha_0^2 \omega_0 d_0^3 - \varepsilon \omega_0 d_0 + (1-k)c_0 + k - \tfrac{3}{4}\mu \alpha_0^2 (c_0^3 + c_0 d_0^2 - 1) = 0,$$

$$-\omega_0^2 d_0 - \tfrac{1}{4}\varepsilon \alpha_0^2 \omega_0 d_0^2 c_0 - \tfrac{1}{4}\varepsilon \alpha_0^2 \omega_0 c_0^3 + \varepsilon \omega_0 c_0 + (1-k)d_0 - \tfrac{3}{4}\mu \alpha_0^2 (c_0^2 d_0 + d_0^3) = 0.$$

The above set of algebraic equations have only two real solutions,

$$\alpha_0 = 2, \quad \omega_0 = 1, \quad c_0 = 1, \quad d_0 = 0, \tag{39}$$

and

$$\alpha_0 = 2, \quad \omega_0 = \sqrt{1-2k-6\mu}, \quad c_0 = -1, \quad d_0 = 0. \tag{40}$$

It is found that

$$c_m = 0, \quad d_m = 0, \quad m \geq 1,$$

in the two cases mentioned above. So, there are a periodic solution given by $c=c_0=1$, $d=0$, and a periodic solution given by $c=c_0=-1$, $d=0$, respectively. In case of $c_0=1$ and $d_0=0$, the two periodic solutions have the same amplitude and the same frequency without phase difference, and therefore are "in phase." In case of $c_0=-1$ and $d_0=0$, the two periodic solutions have the same frequency and the same amplitude but with a phase difference π, and thus are "out of phase." We will show this point later in details. Note that, in case of in-phase oscillations ($c_0=1$), both of ω_0 and α_0 are independent of the two couple parameters k and μ. However, in case of out-of-phase oscillations ($c_0=-1$), the initial guess ω_0 is a function of k and μ, which indicates that the out-of-phase oscillations exist only when

$$k + 3\mu \leq \tfrac{1}{2}.$$

III. PERIODIC OSCILLATIONS

A. In-phase periodic oscillations

Using (39) as the initial guess, we have the fifth-order homotopy analysis approximation of the frequency ω and amplitude α of the in-phase periodic oscillations,

$$\begin{aligned}\omega = 1 &+ \hbar\left(\tfrac{5}{16} + \tfrac{5}{8}\hbar + \tfrac{5}{8}\hbar^2 + \tfrac{5}{16}\hbar^3 + \tfrac{1}{16}\hbar^4\right)\varepsilon^2 + \hbar^2\left(\tfrac{15}{256} + \tfrac{185}{1536}\hbar + \tfrac{95}{1024}\hbar^2 + \tfrac{13}{512}\hbar^3\right)\varepsilon^4 \\ &+ \hbar^3\left(\tfrac{25}{4096} + \tfrac{475}{49\,152}\hbar + \tfrac{3115}{884\,736}\hbar^2\right)\varepsilon^6 + \hbar^4\left(\tfrac{175}{524\,288} + \tfrac{455}{1\,572\,864}\hbar\right)\varepsilon^8 + \tfrac{63}{8\,388\,608}\hbar^5\varepsilon^{10},\end{aligned} \tag{41}$$

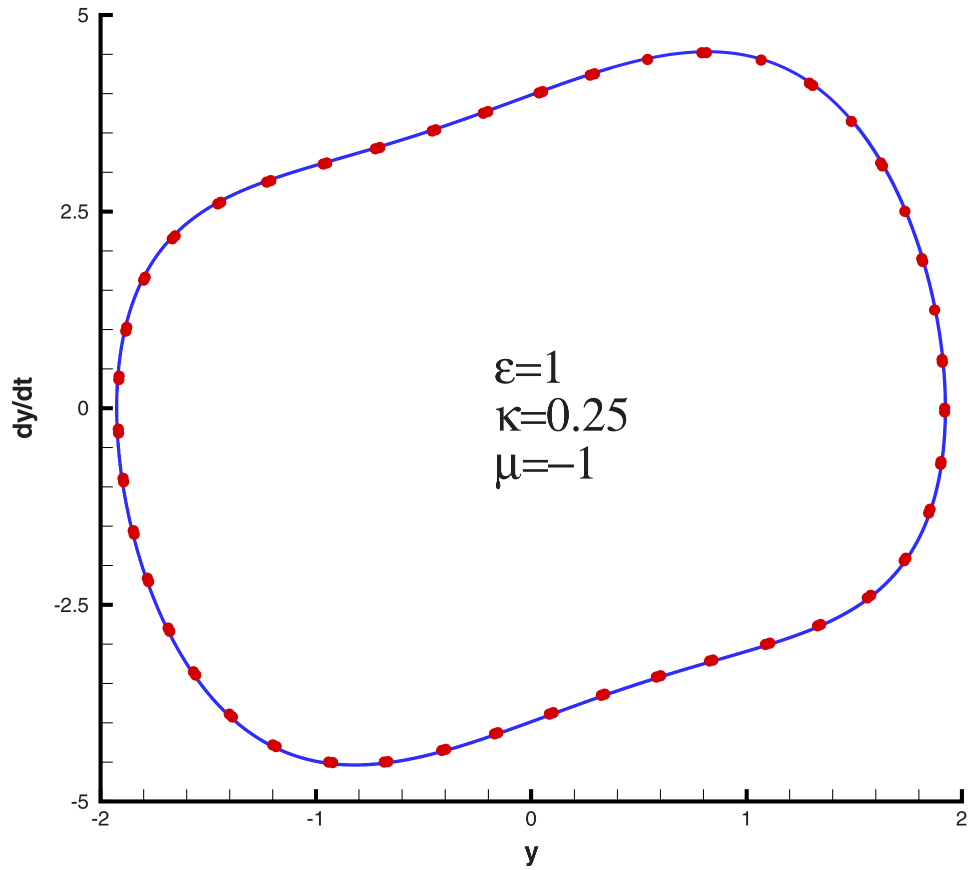

FIG. 2. (Color online) Comparison of the homotopy approximation and numerical result for $\varepsilon=1$, $k=\frac{1}{4}$, $\mu=-1$. Filled circles: tenth-order homotopy approximation when $\hbar=-0.14$; line: numerical result.

$$\alpha = 2 + \hbar^2\left(\frac{5}{48} + \frac{5}{24}\hbar + \frac{5}{32}\hbar^2 + \frac{1}{24}\hbar^3\right)\varepsilon^2 + \hbar^3\left(\frac{5}{384} + \frac{1847}{110\,592}\hbar + \frac{767}{138\,240}\hbar^2\right)\varepsilon^4 + \hbar^4\left(\frac{5}{614} + \frac{571}{1\,105\,920}\hbar\right)\varepsilon^6 + \frac{1}{49\,152}\hbar^5\varepsilon^8. \tag{42}$$

It is found that both the frequency ω and the amplitude α of the in-phase periodic oscillations are *independent* of k and μ. So, do the two corresponding periodic solutions $y(t)$ and $z(t)$. Therefore, the two Van der Pol equations are, in fact, decoupled. Note that these approximations contain the convergence-control parameter $\hbar$, which can be used to control the convergence of ω and α, as pointed by Liao.[8–14] Besides, the so-called homotopy-Padé technique[11,12] can be used to accelerate the convergence of the series result, as shown in Fig. 1. Obviously, as ε increases, the angular frequency ω of the periodic solutions decreases rapidly, but the amplitude increases slightly. These results are exactly the same as a single Van der Pol oscillator, and thus we do not discuss them anymore. Here, we just emphasize that the two coupled Van der Pol equations are decoupled in the case of the in-phase periodic oscillations, even if there exists a nonlinear couple term μ.

B. Out-of-phase periodic oscillations

Using (40) as the initial guess, we obtain the second-order approximations of ω and α of out-of-phase periodic oscillations,

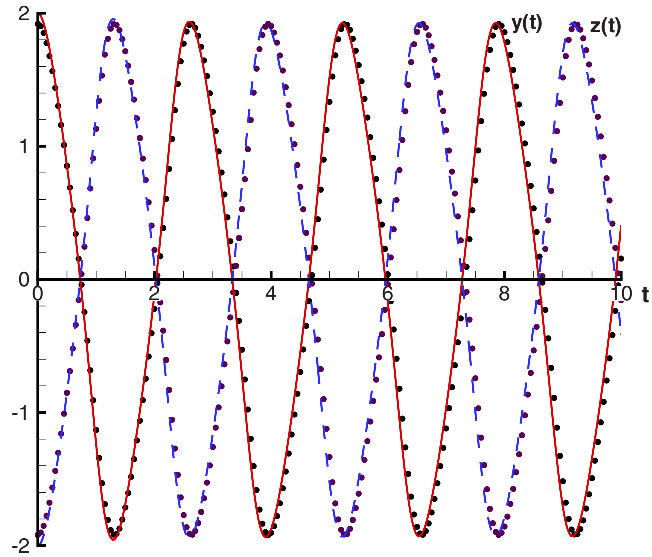

FIG. 3. (Color online) Comparison of HAM approximation time-history curves and numerical results of y and z when $\varepsilon=1$, $k=\frac{1}{4}$, $\mu=-1$. solid line: tenth-order homotopy approximation of $y(t)$ when $\hbar=-0.14$; dash Line: tenth-order homotopy approximation of $z(t)$ when $\hbar=-0.14$.

$$\omega=\frac{1}{512}\frac{\omega_0}{(-1+2k+6\mu)^2}[43\,632\hbar^2\mu^4-\hbar(13\,824+14\,400\hbar+7920\varepsilon^2\hbar-28\,800k\hbar)\mu^3 + (4608k^2\hbar^2-\hbar(4608+4608\hbar+7440\varepsilon^2\hbar)k+2304\hbar+108\varepsilon^4\hbar^2+3720\varepsilon^2\hbar^2+1152\hbar^2 + 2304\varepsilon^2\hbar+18\,432)\mu^2-(2368\varepsilon^2\hbar^2k^2-(12\,288+1536\varepsilon^2\hbar+2368\varepsilon^2\hbar^2+72\varepsilon^4\hbar^2)k+768\varepsilon^2\hbar + 36\varepsilon^4\hbar^2+6144+592\varepsilon^2\hbar^2)\mu-256\varepsilon^2\hbar^2k^3+(384\varepsilon^2\hbar^2+256\varepsilon^2\hbar+2048+12\varepsilon^4\hbar^2)k^2-(2048 + 192\varepsilon^2\hbar^2+256\varepsilon^2\hbar+12\varepsilon^4\hbar^2)k+512+32\varepsilon^2\hbar^2+3\varepsilon^4\hbar^2+64\varepsilon^2\hbar], \tag{43}$$

$$\alpha=2-\frac{\hbar^2\varepsilon^2}{96}(2k-1)-\hbar\left(\frac{1}{2}-k\hbar+\frac{1}{16}\varepsilon^2\hbar+\frac{1}{2}\hbar+\frac{1}{2}\right)\mu+\frac{49}{16}\hbar^2\mu^2, \tag{44}$$

where $\omega_0=\sqrt{1-2k-6\mu}$. So, in the case of out-of-phase periodic oscillations, the two periodic solutions are strongly coupled and influenced by the two couple parameters k and μ. This is essentially different from the in-phase case.

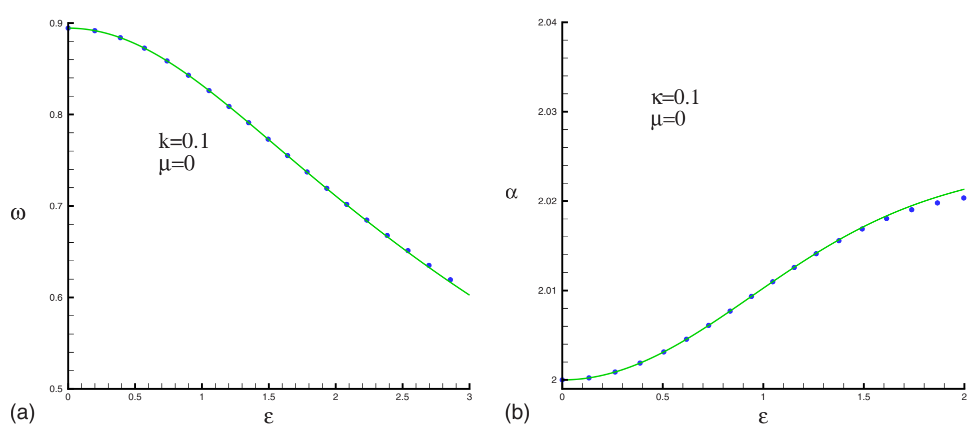

FIG. 4. (Color online) The curve $\varepsilon\sim\omega$ and $\varepsilon\sim\alpha$ by HAM-Padé approximation in the out-of-phase case when $k=0.1$, $\mu=0$. Filled circles: [5,5] HAM-Padé approximation; line: [10,10] HAM-Padé approximation.

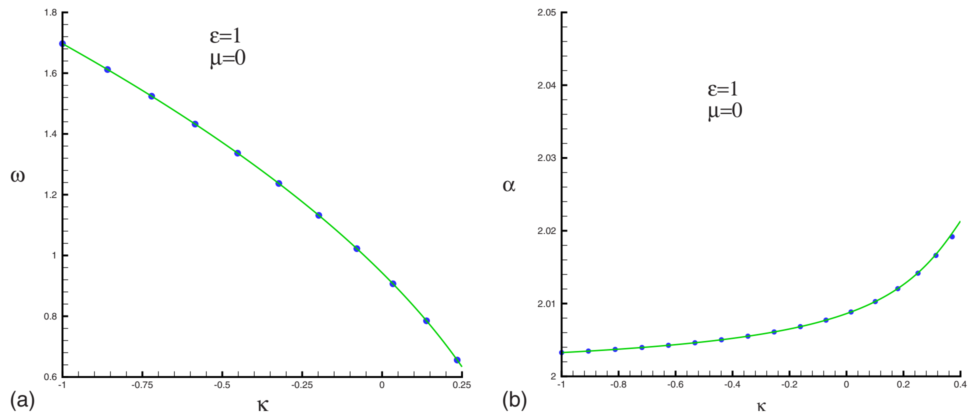

FIG. 5. (Color online) The curve $k \sim \omega$ and $k \sim \alpha$ by HAM-Padé approximation in the out-of-phase case when $\varepsilon=1$, $\mu=0$. Filled circles: [5,5] HAM-Padé approximation; line: [10,10] HAM-Padé approximation.

Our approach can give convergent series in general cases. For example, in case of $\varepsilon=1$, $k=\frac{1}{4}$, $\mu=-1$, our series solution agrees well with numerical ones, as shown in Fig. 2. It is found that the two periodic solutions have indeed a phase difference π, i.e., $y(t)+z(t)=0$, as shown in Fig. 3.

The influence of the physical parameter ε and the two couple parameters k,μ on the out-of-phase periodic oscillations are investigated. For given couple parameters k and μ, the frequency of the coupled periodic solutions decreases monotonously, but the amplitude α increases slightly, when the nonlinearity parameter ε increases, as shown in Fig. 4. For given ε and μ, the frequency decreases monotonously but the amplitude α increases slightly, when linear couple parameter k increases, as shown in Fig. 5. For given ε and k, the frequency also decreases monotonously but the amplitude α increases slightly, when cubic couple parameter μ increases, as shown in Fig. 6. Note that the cubic couple parameter μ has larger influence on the out-of-phase periodic oscillations than the linear couple parameter k, as shown in Figs. 5 and 6.

Therefore, in the case of out-of-phase oscillations, the two couple periodic solutions are strongly influenced by the two couple parameters, and the influence becomes stronger as the two couple parameters k and μ increase. So, we can control and/or adjust the frequency and amplitude of the out-of-phase periodic solutions of the two-coupled Van der Pol equations by means of the

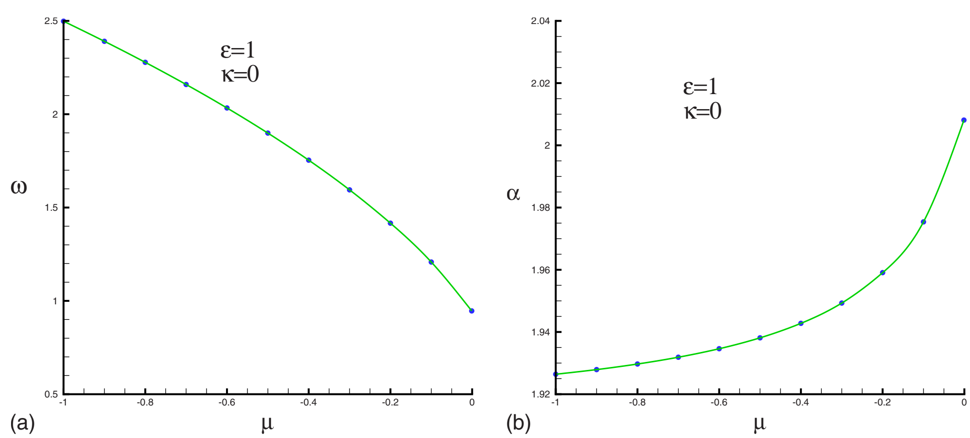

FIG. 6. (Color online) The approximate curve $\mu \sim \omega$ and $\mu \sim \alpha$ by HAM-Padé approximation in the out-of-phase case when $\varepsilon=1$, $k=0$. Filled circles: [4,4] HAM-Padé approximation; line: [5,5] HAM-Padé approximation.

two couple parameters k and μ. However, we cannot change the frequency and amplitude of the in-phase periodic solutions, which are decoupled and have nothing to do with the two couple parameters k and μ.

All above mentioned are under the assumption that the two periodic solutions have the same frequency. In a similar way, we also investigate the case that the frequency of the periodic solution $z(t)$ is integer times of that for $y(t)$, and it is found that no periodic solutions exist in these cases. Therefore, there exist only either in-phase or out-of-phase periodic solutions for the two coupled Van der Pol equations.

IV. CONCLUSION AND DISCUSSION

Nohara and Arimoto[7] proved that the two linearly coupled Van der Pol equations have either in-phase or out-of-phase periodic solutions. In this paper, we consider the periodic solutions of two coupled Van der Pol equations which are connected by a linear and a cubic couple term. The analytic method for strongly nonlinear problems, namely, the HAM, is applied to get series solutions. Different from other analytic techniques, the HAM provides us a convenient way to guarantee the convergence of solutions series. Therefore, using the HAM as an analytic tool, we find out all possible periodic solutions of the two coupled Van der Pol equations. It is found that there exist either in-phase or out-of-phase periodic solutions even for two Van der Pol equations with a cubic couple term. Mathematically, our approach has general meanings.

For the in-phase oscillations, the two periodic solutions are exactly the same, i.e., $y(t)=z(t)$, but decoupled, because the corresponding frequency and amplitude are independent of the two couple parameters k and μ. In this case, it is impossible to adjust the frequency and amplitude of the periodic solutions by means of changing the values of k and μ. However, for the out-of-phase oscillations, the two periodic solutions have the same frequency and the same amplitude but with a phase difference π, i.e., $y(t)+z(t)=0$, and besides are strongly coupled. In this case, it is found that the frequency decreases monotonously but the amplitude α increases slightly, when the two couple parameters increase. So, for the out-of-phase oscillations, we can adjust the frequency and amplitude by changing the two couple parameters k and μ. This conclusion might be useful in practice.

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