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A KIND OF APPROXIMATE SOLUTION TECHNIQUE WHICH DOES NOT DEPEND UPON SMALL PARAMETERS — II. AN APPLICATION IN FLUID MECHANICS

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Abstract—In this paper, the non-linear approximate technique called Homotopy Analysis Method proposed by Liao is further improved by introducing a non-zero parameter into the traditional way of constructing a homotopy. The 2D viscous laminar flow over an infinite flat-plain governed by the non-linear differential equation $f'''(\eta) + f(\eta)f''(\eta)/2 = 0$ with boundary conditions $f(0) = f'(0) = 0$, $f'(+\infty) = 1$ is used as an example to describe its basic ideas. As a result, a family of approximations is obtained for the above-mentioned problem, which is much more general than the power series given by Blasius [*Z. Math. Phys.* **36**, 1(1908)] and can converge even in the whole region $\eta \in [0, +\infty)$. Moreover, the Blasius' solution is only a special case of ours. We also obtain the second-derivative of $f(\eta)$ at $\eta = 0$, i.e. $f''(0) = 0.33206$, which is exactly the same as the numerical result given by Howarth [*Proc. Roy. Soc. London* **A164**, 547 (1938)]. © 1997 Elsevier Science Ltd.

Keywords: non-linear technique, homotopy, HAM, Blasius' flow

1. INTRODUCTION

Perturbation techniques have been widely applied to solve non-linear problems. Unfortunately, all perturbation techniques are based on such an assumption that a small parameter *must* exist. This so-called small parameter assumption greatly restricts applications of perturbation techniques, because many non-linear problems, especially those having strong non-linearity, have no small parameters at all. Moreover, even if there exists such a small parameter, the corresponding perturbation approximations are valid generally only for small values of this parameter and become useless as the value of the parameter increases.

For instance, consider the two-dimensional (2D) viscous laminar flow over an infinite flat-plain governed by a non-linear ordinary differential equation

$$f''' + \tfrac{1}{2}ff'' = 0, \quad \eta \in [0, +\infty), \tag{1.1}$$

with boundary conditions

$$f(0) = f'(0) = 0, \quad f'(+\infty) = 1, \tag{1.2}$$

where the prime denotes the derivatives with respect to η which is defined as

$$\eta = y\sqrt{\frac{U}{vx}},$$

and $f(\eta)$ is related to the streamfunction ψ by

$$f(\eta) = \frac{\psi}{\sqrt{vUx}}.$$

Here, U is the velocity at infinity, v is the kinematic viscosity coefficient, x and y are the two independent coordinates. Note that (1.1) is a special case of the so-called Falkner–Skan

Contributed by P. Hagedorn.

equation

$$f''' + \alpha f f'' + \beta(1 - f'^2) = 0,$$

proposed by Falkner and Skan [3], which was studied by Hartree [4] in 1937 for the physical problem of boundary layers in plates and also by Howarth [2] in 1938. Moreover, Schroeder [5] and Görtler [6] also studied such a problem numerically. For details, please refer to Schlichting [7, 8], Dewey and Gross [9], Hiemenz [10], and Smith and Cebeci [11].

By means of Taylor's series, in 1908 Blasius [1] gave a solution of the non-linear equation (1.1) in the form of a power series

$$f(\eta) = \sum_{k=0}^{+\infty} \left(-\frac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2}, \tag{1.3}$$

where

$$A_k = \begin{cases} 1 & (k = 0 \cup k = 1), \\ \sum_{r=0}^{k-1} \binom{3k-1}{3r} A_r A_{k-r-1} & (k \geqslant 2), \end{cases} \tag{1.4}$$

with the definition

$$\binom{m}{n} = \frac{m!}{n!(m-n)!} = C_m^n.$$

Note that the expression (1.3) is not closed, because $\sigma = f''(0)$ is unknown. For large η, Blasius [1] gave another approximation of $f(\eta)$. Then, by means of matching his two different approximations at a proper point, Blasius obtained the result $\sigma = 0.332$. Later, Howarth [2] gave a more accurate value of σ, i.e. $\sigma = 0.33206$, by means of numerical techniques. However, even setting $\sigma = 0.33206$ in (1.3), the convergence radius of the power series (1.3) is still finite and the power series (1.3) is valid only in a small region $0 \leqslant \eta < \rho_0 = 5.690$, as shown in Fig. 1.

It is interesting that, supposing the value of $\sigma = f''(0)$ is known, one can also obtain the power series (1.3) by means of perturbation techniques [12–14] in the following way. First of all, we introduce a small parameter ε and consider such a non-linear equation

$$f''' + \frac{\varepsilon}{2} f f'' = 0, \quad \eta \in [0, +\infty), \tag{1.5}$$

with boundary conditions

$$f(0) = f'(0) = 0, \quad f''(0) = \sigma. \tag{1.6}$$

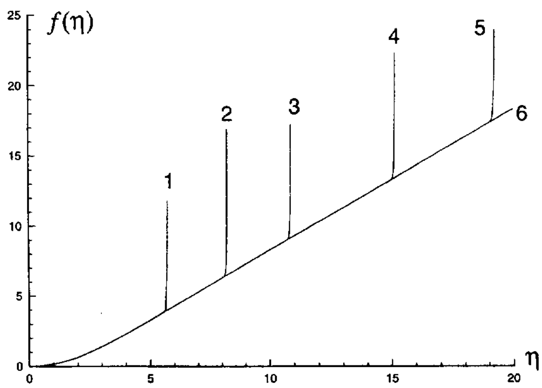

Fig. 1. Comparisons of the numerical solution with the approximations (2.9) under different values of $\hbar$ in the order $m = 100$. Curve 1: $\hbar = -1$ (Blasius' power series); Curve 2: $\hbar = -1/2$; Curve 3: $\hbar = -1/4$; Curve 4: $\hbar = -1/10$; Curve 5: $\hbar = -1/20$; Curve 6: numerical result given by Howarth ([2]).

Then, we *suppose* that $f(\eta)$ could be expressed as

$$f(\eta) = \sum_{k=0}^{+\infty} \varepsilon^k \tilde{f}_k(\eta). \tag{1.7}$$

Substituting (1.7) into (1.5) and (1.6), we obtain a series of linear equations:
zeroth-order equation:

$$\tilde{f}_0''' = 0, \tag{1.8}$$

$$\tilde{f}_0(0) = \tilde{f}_0'(0) = 0, \quad \tilde{f}_0''(0) = \sigma. \tag{1.9}$$

first-order equation:

$$\tilde{f}_1''' = -\tfrac{1}{2}\tilde{f}_0\tilde{f}_0'', \tag{1.10}$$

$$\tilde{f}_1(0) = \tilde{f}_1'(0) = \tilde{f}_1''(0) = 0. \tag{1.11}$$

second-order equation:

$$\tilde{f}_2''' = -\tfrac{1}{2}(\tilde{f}_0\tilde{f}_1'' + \tilde{f}_1\tilde{f}_0''), \tag{1.12}$$

$$\tilde{f}_2(0) = \tilde{f}_1'(0) = \tilde{f}_2''(0) = 0. \tag{1.13}$$

Solving above linear equations one after another in order, one obtains

$$\tilde{f}_0(\eta) = \frac{\sigma}{2}\eta^2, \tag{1.14}$$

$$\tilde{f}_1(\eta) = -\frac{\sigma^2}{240}\eta^5, \tag{1.15}$$

$$\tilde{f}_2(\eta) = \frac{11}{161{,}280}\sigma^3\eta^8. \tag{1.16}$$

Substituting the above results into (1.7) and then setting $\varepsilon = 1$, we obtain exactly the same power series as (1.3) given by Blasius [1] in 1908. However, as mentioned above, this power series converges only in a small region even if we use the accurate enough numerical result $f''(0) = 0.33206$ given by Howarth [2], as shown in Fig. 1. This seems reasonable, because perturbation techniques are based on small parameter assumption so that perturbation approximations are valid generally only for small parameters or small variables. This is also the main reason why Blasius had to give another approximation of $f(\eta)$ for large η. So, it seems necessary to give a kind of analytical solution of (1.1) and (1.2), which should be *uniformly* valid for both small and large values of η.

Liao [15–17] has proposed a new kind of non-linear analytical technique, namely Homotopy Analysis Method (HAM). HAM is quite different from perturbation techniques, because it is based on homotopy in topology [18, 19] and does not depend upon small parameters at all. Owing to this reason, HAM can be used to solve more non-linear problems, even including those whose governing equations and boundary conditions do not contain any small parameters at all. Moreover, it can give accurate enough approximations which are uniformly valid for both small and large parameters or variables, as mentioned by Liao [15–17].

This paper is the continuation of the author's work described in [15–17]. In this paper, the proposed Homotopy Analysis Method is further greatly improved by introducing a non-zero parameter into the classical way of constructing a homotopy. Its basic ideas are described in detail by solving the above-mentioned, famous non-linear equations (1.1) and (1.2) in fluid mechanics.

2. BASIC IDEAS OF HOMOTOPY ANALYSIS METHOD

First of all, we construct such a family of equations, called *zeroth-order deformation equation*

$$(1-p)[F'''(\eta,\hbar;p) - \tilde{f}_0'''(\eta)] = p\hbar[F'''(\eta,\hbar;p) + \tfrac{1}{2}F(\eta,\hbar;p)F''(\eta,\hbar;p)],$$
$$p \in [0,1], \quad \hbar \neq 0, \quad \eta \in [0,+\infty), \tag{2.1}$$

with corresponding boundary conditions at $\eta = 0$, i.e.

$$F(0, \hbar; p) = F'(0, \hbar; p) = 0, \quad F''(0, \hbar; p) = \sigma,$$
$$p \in [0, 1], \quad \hbar \neq 0, \tag{2.2}$$

where $p \in [0, 1]$ is an embedding parameter, $f_0(\eta) = \sigma\eta^2/2$ is an initial approximation which satisfies the boundary conditions (1.6), $\sigma = f''(0)$ is the second-order derivative of $f(\eta)$ at $\eta = 0$, and the prime denotes derivatives with respect to η. What we should emphasize here is the newly introduced non-zero real number $\hbar$ ($\hbar \neq 0$), called *homotopy parameter*, which was not used in [15–17]. Obviously, at $p = 0$, we have by (2.1) and (2.2) that $F(\eta, \hbar; 0) = f_0(\eta) = \sigma\eta^2/2$. Moreover, at $p = 1$, the solution of equations (2.1) and (2.2) is exactly the same as that of (1.1) and (1.6) for all values of $\hbar$ except $\hbar = 0$, so that we have $F(\eta, \hbar; 1) = f(\eta)$. It means that, for any fixed non-zero value of $\hbar$ ($\hbar \neq 0$), $F(\eta, \hbar; p)$ is a homotopy with the embedding parameter $p \in [0, 1]$. Although the homotopies $F(\eta, \hbar; p): f_0(\eta) \cong f(\eta)$ have the same start-point $f_0(\eta)$ and the same end-point $f(\eta)$, their traces $F(\eta, \hbar; p)$ are dependent upon the parameter $\hbar$ and might be quite different. Therefore, (2.1) and (2.2) in fact construct a *family* of homotopies, in place of only one kind of homotopy which is traditionally used. Certainly, there should exist better homotopies or even the best one among them, which should give better approximations or even the best approximation, respectively.

Differentiating (2.1) and (2.2) m times with respect to p and then setting $p = 0$, we obtain the corresponding *mth-order deformation equation* at $p = 0$, i.e.

$$(f_0^{[m]})''' = g_m(\eta, \hbar), \quad \eta \in [0, +\infty), \quad \hbar \neq 0, \tag{2.3}$$

with corresponding boundary conditions:

$$f_0^{[m]}(0, \hbar) = (f_0^{[m]})'(0, \hbar) = (f_0^{[m]})''(0, \hbar) = 0, \tag{2.4}$$

where

$$g_1(\eta, \hbar) = \hbar(f_0''' + \tfrac{1}{2} f_0 f_0''), \tag{2.5}$$

$$g_m(\eta, \hbar) = m\left\{(1 + \hbar)(f_0^{[m-1]})''' + \tfrac{1}{2}\hbar \sum_{k=0}^{m-1} \binom{m-1}{k} f_0^{[k]} (f_0^{[m-1-k]})''\right\} (m \geqslant 2), \tag{2.6}$$

with the definition

$$f_0^{[m]}(\eta, \hbar) = \left.\frac{\partial^m F(\eta, \hbar; p)}{\partial p^m}\right|_{p=0}, \tag{2.7}$$

called the *mth-order deformation derivative* at $p = 0$. Integrating (2.3) three times about η and then determining the corresponding integration constants by the conditions (2.4), we can easily obtain $f_0^{[m]}(\eta, \hbar)(m \geqslant 1)$, especially by means of the widely-applied software *MATHEMATICA* (see [20]). To the author's surprise, the mth-order approximation of $f(\eta)$, i.e.

$$f_m(\eta, \hbar) = f_0(\eta) + \sum_{k=1}^{m} \frac{f_0^{[k]}(\eta, \hbar)}{k!}, \quad \eta \in [0, +\infty), \quad \hbar \neq 0, \tag{2.8}$$

can be simply described as

$$f_m(\eta, \hbar) = \sum_{k=0}^{m} \left[(-\tfrac{1}{2})^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2}\right] \Phi_{m,k}(\hbar),$$
$$\eta \in [0, +\infty), \quad \hbar \neq 0, \tag{2.9}$$

where the real function $\Phi_{m,n}(\hbar)$ is defined by

$$\Phi_{m,n}(\hbar) = \begin{cases} 0 & (n > m), \\ (-\hbar)^n \sum_{k=0}^{m-n} \binom{m}{m-n-k} \binom{n+k-1}{k} \hbar^k & (1 \leqslant n \leqslant m), \\ 1 & (n \leqslant 0). \end{cases} \tag{2.10}$$

We call $\Phi_{m,n}(\hbar)$ the *approaching function*. It is easy to prove rigorously that the approaching function $\Phi_{m,n}(\hbar)$ has the following fundamental properties:

(a) for integer m ($m \geqslant 0$) and n

$$\langle 1 \rangle \quad \Phi_{m,n}(\hbar) = \begin{cases} 1 & (n \leqslant 0), \\ 0 & (n > m), \end{cases} \tag{2.11}$$

(b) for integer m ($m \geqslant 0$) and n ($n \leqslant m$)

$$\langle 2 \rangle \quad \Phi_{m,n}(-1) = 1, \tag{2.12}$$

(c) for positive integer n ($n \geqslant 1$)

$$\langle 3 \rangle \quad \lim_{m \to +\infty} \Phi_{m,n}(\hbar) = 1 \quad (-2 < \hbar < 0), \tag{2.13}$$

$$\langle 4 \rangle \quad \lim_{m \to +\infty} |\Phi_{m,n}(\hbar)| = +\infty \quad (\hbar > 0 \cup \hbar \leqslant -2), \tag{2.14}$$

(d) for finite, positive integer r ($r \geqslant 0$)

$$\langle 5 \rangle \quad \lim_{m \to +\infty} |\Phi_{m,m-r}(\hbar)| = \begin{cases} 0 & |\hbar| < 1, & \\ +\infty & |\hbar| > 1, & \\ +\infty & \hbar = 1, & r > 0, \\ 1 & \hbar = 1, & r = 0, \\ 1 & \hbar = -1, & r \geqslant 0. \end{cases} \tag{2.15}$$

Moreover, the approaching function $\Phi_{m,n}(\hbar)$ has many other properties such as

$$\Phi'_{m,n}(\hbar) = (-1)^n n \binom{m}{n} \hbar^{n-1}(1+\hbar)^{m-n} \quad (1 \leqslant n \leqslant m),$$

$$\Phi_{m+1,n}(\hbar) - \Phi_{m,n}(\hbar) = \binom{m}{n-1}(-\hbar)^n(1+\hbar)^{m-n+1} \quad (1 \leqslant n \leqslant m),$$

and so on. Considering the length of this paper and the nature of the journal, the abstract mathematical proofs of these properties are not presented here.

Note that we have now a *family* of approximations (2.9). Certainly, some approximations should be better than others. And there might exist even the best approximation among them. According to the property (2.12) of the approaching function $\Phi_{m,n}(\hbar)$, the power series (2.9) when $\hbar = -1$ is the same as the Blasius' power series (1.3), which means that Blasius' power series is a member of the family (2.9), so that it is only a special case of this family of approximations. It is interesting that, when $-1 < \hbar < 0$, the power series (2.9) is valid in larger regions, as shown in Fig. 1, where $\sigma = 0.33206$ is used. Note that, as $|\hbar|$ ($-1 \leqslant \hbar < 0$) becomes smaller and smaller, the convergent region of the power series (2.9) becomes larger and larger. Therefore, simply multiplying the Blasius' power series (1.3) by the approaching function $\Phi_{m,n}(\hbar)$ one after another in order, we obtain, when $-1 < \hbar < 0$, much better approximations than (1.3), which is only a special case of (2.9) when $\hbar = -1$. This indicates that the proposed non-linear analytical method, namely Homotopy Analysis Method (HAM), has indeed great potential.

Note that $f''(0) = 0.33206$ is used in (2.9). In fact, using the condition $f'(+\infty) = 1$, we can obtain the value of $f''(0)$ by means of solving the following algebraic equation for a proper value of $\hbar = \hbar_0$ at a proper point $\eta = \eta_0$ far enough from point $\eta = 0$

$$\left.\frac{\partial f_m(\eta,\hbar)}{\partial \eta}\right|_{\eta=\eta_0,\hbar=\hbar_0} = \sum_{k=0}^{m} \left[\left(-\frac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+1)!} \eta_0^{3k+1}\right] \Phi_{m,k}(\hbar_0) = 1. \tag{2.16}$$

For large enough m and small enough $|\hbar_0|$ ($-1 < \hbar_0 < 0$), the above equation at five different points in the region $\eta \in [8, 9]$ gives the same value $\sigma = f''(0) = 0.33206$, which is exactly the same as the numerical result given by Howarth [2]. The detailed numerical results and the corresponding numerical parameters are given in Table 1. Note that we use here only the expression (2.9), but Blasius had to match two different approximations which are valid for small and large η, respectively.

Table 1. Numerical values of $f''(0)$ given by solving (2.16) under different orders m

Order m	$\hbar_0 = -\frac{1}{10}$, $\eta_0 = 8$	$\hbar_0 = -\frac{1}{10}$, $\eta_0 = 8.25$	$\hbar_0 = -\frac{1}{12}$, $\eta_0 = 8.50$	$\hbar_0 = -\frac{1}{12}$, $\eta_0 = 8.75$	$\hbar_0 = -\frac{1}{12}$, $\eta_0 = 9.00$
10	0.31222	0.30968	0.30536	0.30223	0.29967
20	0.32881	0.32928	0.32614	0.32690	0.32743
30	0.33146	0.33138	0.33074	0.33061	0.33045
40	0.33185	0.33184	0.33152	0.33149	0.33149
50	0.33200	0.33200	0.33188	0.33189	0.33190
60	0.33205	0.33205	0.33200	0.33201	0.33201
70	0.33206	0.33205	0.33204	0.33204	0.33204
80	0.33206	0.33206	0.33205	0.33205	0.33205
90	0.33206	0.33206	0.33206	0.33206	0.33206
100	0.33206	0.33206	0.33206	0.33206	0.33206

The Blasius' power series (1.3) converges in a small region $-\rho_0 < \eta < \rho_0$, where $\rho_0 = 5.690$. However, our numerical computations indicate that the generalized power series (2.9) converges in the region

$$-\rho_0 < \eta < \rho_0 \left[\frac{2}{|\hbar|} - 1\right]^{1/3} \quad (-2 < \hbar < 0), \tag{2.17}$$

which becomes larger and larger, as $|\hbar|$ $(-2 < \hbar < 0)$ becomes smaller and smaller, as shown in Fig. 1, where $\rho_0 = 5.690$ is the convergence radius of the Blasius' power series (1.3). It means that the power series (2.9) may converge in the *whole* region $\eta = [0, +\infty)$ as $|\hbar|$ $(-2 < \hbar < 0)$ tends to zero! Moreover, owing to (2.12), the Blasius' power series (1.3) is only a special case of (2.9) at $\hbar = -1$. Note that, when $\hbar = -2$, the power series (2.9) converges in the smallest region $-\rho_0 < \eta < 0$. When $\hbar = -1$, it converges in the region $-\rho_0 < \eta < \rho_0$, which is the same as that of the Blasius' power series (1.3). However, as $|\hbar|$ $(-2 < \hbar < 0)$ tends to zero, the power series (2.9) converges in the largest region $-\rho_0 < \eta < +\infty$. Therefore, the Blasius' power series (1.3) is only a common member of the family of the power series (2.9): it is neither the best nor the worst, but simply a quite ordinary one. Hence, the power series (2.9) is much more general than the Blasius' power series (1.3).

Note that we use in this paper such a linear operator

$$L(F) = F''' = \frac{\partial^3 F}{\partial \eta^3} \tag{2.18}$$

to construct the zeroth-order deformation equation (2.1), and this leads to the simple, elegant expression (2.9) whose convergent region is a function of $\hbar$ $(-2 < \hbar < 0)$. However, by means of Homotopy Analysis Method (HAM), we have quite large freedom to select other linear operators. For instance, if we use a more general linear operator such as

$$L(F) = \frac{\partial^3 F}{\partial \eta^3} + \gamma \frac{\partial^2 F}{\partial \eta^2} \quad (\gamma \geqslant 0), \tag{2.19}$$

we can also obtain a family of approximations even more general than (2.9), which, although much more complex, can converge in the whole region $\eta = (0, +\infty)$, and whose second-order derivative at $\eta = 0$ converges to exactly 0.32206. Note that the linear operator (2.18) used in equation (2.1) is only a special case of the linear operator (2.19) when $\gamma = 0$ so that (2.9) itself is only a member of the even larger family given by the more general linear operator (2.19) in a similar way. We will discuss this point in detail in the near future.

As the last part of this section, the author would like to point out that the power series (2.9) with the corresponding convergence radius (2.17) is only a special case of an abstract mathematical theorem described in the Appendix, called generalized Taylor's theorem, which has been rigorously proved by the author from the view-point of pure mathematics. Considering the length of this paper, its proof is omitted here but will appear soon in a mathematical journal.

3. CONCLUSIONS

In this paper, the Homotopy Analysis Method (HAM) proposed by Liao [15–17] is further improved by introducing a non-zero parameter $\hbar$, called the *homotopy parameter*, into the traditional way of constructing a homotopy. The 2D viscous laminar flow over an infinite flat-plain governed by the non-linear equation (1.1) with boundary conditions (1.2) is used as an example to describe its basic ideas. As a result, we obtain a *family* of approximations (2.9) that converge when $\hbar \in (-1, 0)$ in a larger region than that of the power series (1.3) given by Blasius [1]. It is quite interesting that the power series (2.9) is much more general than Blasius' power series (1.3), which is only a special case of the family (2.9) at $\hbar = -1$, because the Blasius' power series (1.3) is neither the worst nor the best, but simply just a quite ordinary example of the family (2.9). Moreover, the convergence region of the power series (2.9) becomes larger and larger, as $|\hbar|$ $(-2 < \hbar < 0)$ becomes smaller and smaller, as shown in Fig. 1. And the power series (2.9) may converge in the *whole* region $\eta \in [0 + \infty)$ as $\hbar$ $(-2 < \hbar < 0)$ tends to zero. Note that, using the power series (2.9) and the boundary condition $f'(+\infty) = 1$, we also obtain $f''(0) = 0.33206$ which is exactly the same as the numerical result given by Howarth [2].

Lastly, we should emphasize that in this paper we do *not* use the so-called small parameter assumption at all, which is however *absolutely necessary* for perturbation techniques. Even so, we still obtain much more general and even much better approximations than Blasius' power series (1.3). The newly introduced non-zero parameter $\hbar$ really brings us a lot of new, interesting things. And the approaching function $\Phi_{m,n}(\hbar)$ defined by (2.10) seems to have deep meanings which might change some basic concepts of pure mathematics about power series, if we consider (2.9) as a generalized Taylor's expression of $f(\eta)$ at $\eta = 0$, as described in the Appendix. All of these indicate that the proposed new non-linear analytical technique, namely Homotopy Analysis Method (HAM), has indeed great potential and deserves further research and applications.

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APPENDIX: GENERALIZED TAYLOR'S THEOREM

Let $\alpha \geqslant 1$, $\beta \geqslant 0$ and $\gamma \geqslant 0$ be integers, and

$$\sum_{k=0}^{+\infty} \frac{g^{(\alpha k+\beta)}(t_0)}{(\alpha k+\beta)!}(t-t_0)^{\alpha k+\beta}$$

denote the classical Taylor's series of the real function $g(t)$ at $t = t_0$ which converges in the region $|t - t_0| < \rho$, where $g^{(\alpha k+\beta)}(t_0)$ $(k \geqslant 0)$ is the $(\alpha k + \beta)$th-order derivative of $g(t)$ at $t = t_0$. Define

$$\mu = \lim_{k\to+\infty} \frac{g^{(\alpha k+\alpha+\beta)}(t_0)}{(\alpha k+\alpha+\beta)!}\frac{(\alpha k+\beta)!}{g^{(\alpha k+\beta)}(t_0)}.$$

Then, the generalized Taylor's series of the real function $g(t)$ at $t = t_0$, i.e.

$$\lim_{m\to+\infty} \sum_{k=0}^{m} \left[\frac{g^{(\alpha k+\beta)}(t_0)}{(\alpha k+\beta)!}(t-t_0)^{\alpha k+\beta}\right]\Phi_{m,k-\gamma}(\hbar) \quad (-2 < \hbar < 0, \gamma \geqslant 0), \tag{A.1}$$

converges in the region $t \in D$, where $D \subset (-\infty, +\infty)$ is as follows

(A) in case $\mu < 0$ *and* $\alpha = 2k + 1$ $(k = 0, 1, 2, 3, \ldots)$:

$$D = \left\{t: -\rho < t - t_0 < \rho\left[\frac{2}{|\hbar|} - 1\right]^{1/\alpha}, \ -2 < \hbar < 0\right\};$$

(B) in case $\mu < 0$ and $\alpha = 2k$ $(k = 1, 2, 3, \ldots)$:

$$D = \left\{t: |t - t_0| < \rho\left[\frac{2}{|\hbar|} - 1\right]^{1/\alpha}, \ -2 < \hbar < 0\right\};$$

(C) in case $\mu > 0$ and $\alpha = 2k + 1$ $(k = 0, 1, 2, 3, \ldots)$:

$$D = \left\{t: -\rho\left[\frac{2}{|\hbar|} - 1\right]^{1/\alpha} < t - t_0 < \rho, \ -2 < \hbar < 0\right\};$$

(D) in case $\mu > 0$ and $\alpha = 2k$ $(k = 1, 2, 3, \ldots)$:

$$D = \{t: |t - t_0| < \rho, \ -2 < \hbar < 0\}.$$

Moreover, it holds in case $-2 < \hbar < 0$ for any a finite integer $N \geqslant 0$ that

$$\lim_{m\to+\infty} \sum_{k=0}^{N} \left[\frac{g^{(\alpha k+\beta)}(t_0)}{(\alpha k+\beta)!}(t-t_0)^{\alpha k+\beta}\right]\Phi_{m,k-\gamma}(\hbar) = \sum_{k=0}^{N} \frac{g^{(\alpha k+\beta)}(t_0)}{(\alpha k+\beta)!}(t-t_0)^{\alpha k+\beta}.$$

And in case $\hbar = -1$, (A.1) is exactly the classical Taylor's series of $g(t)$ at $t = t_0$. Here, the real function $\Phi_{m,n}(\hbar)$ is defined by

$$\Phi_{m,n}(\hbar) = \begin{cases} 0 & (n > m), \\ (-\hbar)^n \sum_{k=0}^{m-n} \binom{m}{m-n-k}\binom{n+k-1}{k}\hbar^k & (1 \leqslant n \leqslant m), \\ 1 & (n \leqslant 0). \end{cases}$$